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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVES AND RULED SURFACES

BY

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PREFACE.

In the geometrical investigations of the last century, one of the most fundamental distinctions has been that between metrical and projective geometry. It is a curious fact that this classification seems to have given rise to another distinction, which is not at all justified by the nature of things. There are certain properties of curves, surfaces, etc., which may be deduced for the most general configurations of their kind, depending only upon the knowledge that certain conditions of continuity are fulfilled in the vicinity of a certain point. These are the so-called infinitesimal properties and are naturally treated by the methods of the differential calculus. The curious fact to which we have referred is that, but for rare exceptions, these infinitesimal properties have been dealt with only from the metrical point of view. Projective geometry, which has made such progress in the course of the century, has apparently disdained to consider the infinitely small parts into which its configurations may be decomposed. It has gained the possibility of making assertions about its configurations as a whole, only by limiting its field to the consideration of algebraic cases, a restriction which is unnecessary in differential geometry.

Between the metrical differential geometry of Monge and Gauss, and the algebraic projective geometry of Poncelet and Plücker, there is left, therefore, the field of projective differential geometry whose nature partakes somewhat of both. The theorems of this kind of geometry are concerned with projective properties of the infinitesimal elements. As in the ordinary differential geometry, the process of integration may lead to statements concerning properties of the configuration as a whole. But, of course, such integration is possible only in special cases. Even with this limitation, however, which lies in the nature of things, the field of projective differential geometry is so rich that it seems well worth while to cultivate it with greater energy than has been done heretofore.

But few investigations belonging to this field exist. The most important contributions are those of Halphen, who has developed an admirable theory of plane and space curves from this point of view. The author has, in the last few years, built up a projective differential geometry of ruled surfaces. In this book we shall confine ourselves to the consideration of these simplest configurations. If time and strength permit, a general theory of surfaces will follow.

In presenting the theories of Halphen, I have nevertheless followed my own methods, both for the sake of uniformity and simplicity. In all cases, I have attempted to indicate clearly those results which are due

to him or to other authors. The general method of treatment and the results which are not specifically attributed to others are, so far as I am aware, due to me. The theory of ruled surfaces has been developed by me in a series of papers, published principally in the Transactions of the American Mathematical Society, beginning in 1901. I have thought it unnecessary to refer to them in detail, the treatment here given being in many respects preferable to that of the original papers. In particular, some errors have been corrected; I hope that no serious mistakes have been allowed to pass over into the present work. To finish these personal remarks, I may add that Chapter II contains a number of important additions to the theory as generally presented, without which it would lack rigor and completeness. The canonical development of Chapter III has also been added by the author.

The examples collected at the end of each chapter are of two kinds. Some of them are mere exercises. Some, however, (those marked with an asterisk), are of a very different nature. They are essentially suggestions for such further investigations, as appear to me to be of promise and importance. I have, also in the body of the book, taken the privilege of pointing out further problems which seem to be of interest. Many others will readily suggest themselves. It is my sincere wish that these suggestions may be helpful toward a further development of this fascinating subject.

The instructor in an American University finds his time fully occupied by other things besides the advancement of Science. The Carnegie Institution of Washington, in recognition of this fact, makes it a part of its policy to give a certain number of men the opportunity to devote all of their time and energy to research. For two years I have had the honor and the good fortune of finding my efforts aided and encouraged by the support of this magnificent institution. Without this aid, the present work would not have seen the light of day for several years. I take this opportunity to express to the Carnegie Institution my fullest gratitude for its help, and for the generosity with which it has left me free to act and move, unfettered by unnecessary conditions and regulations.

In these last two years, I have had occasion to make use of libraries at Göttingen, Paris, Cambridge and Rome. For these privileges I am indebted to Professors Klein, Darboux, Forsyth and Castelnuovo, who met my wishes with the greatest of courtesy. It remains for me to express my thanks to the publishers B. G. Teubner, whose enterprise is a household word in the mathematical world. For their sake, as well as for my own, I hope that this little book may prove to be a success.

Rome, March 2[d], 1905.

E. J. WILCZYNSKI.

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INTRODUCTION.

THE FUNDAMENTAL THEOREMS OF LIE'S THEORY OF CONTINUOUS GROUPS.

In the theory to which this work is devoted, *Lie*'s theory of continuous groups plays a fundamental part. It seems advisable, therefore, to give a brief account of this important subject; we shall not, however, attempt to do more than to give a clear statement of those ideas and theorems belonging to *Lie*'s theory, which will be found useful later on, without insisting upon the proofs.[1])

The n equations

$$x_i' = f_i(x_1, x_2, \ldots x_n), \quad (i = 1, 2, \ldots n),$$

are said to determine a *transformation* of $x_1, \ldots x_n$ into $x_1', \ldots x_n'$, if they can be solved for $x_1, \ldots x_n$, i. e. if the Jacobian

$$\left| \frac{\partial f_i}{\partial x_k} \right|, \quad (i, k = 1, 2, \ldots n)$$

does not vanish identically.

The equations may contain r arbitrary constants $a_1, \ldots a_r$, so that they may be written

$$(1) \qquad x_i' = f_i(x_1, \ldots x_n;\ a_1, \ldots a_r), \quad (i = 1, 2, \ldots n),$$

in which case they represent not merely a single transformation, but a *family*. The r constants are said to be *essential*, if it is impossible to find combinations $A_1, \ldots A_s$ less than r in number such that $x_1', \ldots x_n'$ appear as functions of $x_1, \ldots x_n$ and $A_1, \ldots A_s$ only. The family may then be said to contain ∞^r transformations. It will be called an *r parameter family*.

Let the r parameters in (1) be essential. After having made the transformation (1) converting $(x_1, \ldots x_n)$ into $(x_1', \ldots x_n')$, let us make another transformation of the same family

$$(2) \qquad x_i'' = f_i(x_1', \ldots x_n';\ b_1, \ldots b_r), \quad (i = 1, 2, \ldots n)$$

which converts $(x_1', \ldots x_n')$ into $(x_1'', \ldots x_n'')$.

1) The general theory has been made easily accessible by the lucid treatment in *Lie-Engel*, Theorie der Transformationsgruppen. Teubner. Leipzig 1888—93. Cf. especially vol. 1. A shorter account is given in *Lie-Scheffers*, Vorlesungen über kontinuierliche Gruppen etc. Teubner. Leipzig 1893. See also *Campbell*, Introductory treatise on Lie's Theory, etc., Clarendon Press, Oxford, 1903.

If, now, we eliminate x_i' between (1) and (2), i. e. if we make directly the transformation which converts $(x_1, \ldots x_n)$ into $(x_1'', \ldots x_n'')$, *it may happen* that the resulting equations are again of the same form as (1) and (2), i. e of the form

$$(3) \qquad x_i'' = f_i(x_1, \ldots x_n;\ c_1, \ldots c_r), \quad (i = 1, 2, \ldots n),$$

where $c_1, \ldots c_r$ are certain functions of a_k, b_k,

$$(4) \qquad c_k = \varphi_k(a_1, \ldots a_r;\ b_1, \ldots b_r), \quad (k = 1, 2, \ldots r).$$

In that case the transformations (1) are said to form a *group*. Moreover to describe these equations more completely we may speak of this group as an *r parameter group in n variables*.

Let equations (1) represent such an r parameter group, and assume that it contains the inverse of each of its transformations. It will, then, contain also the *identical transformation*, i. e. a certain transformation, corresponding to the parameters $a_1^0, \ldots a_r^0$, which reduces to

$$x_i' = x_i, \quad (i = 1, 2, \ldots n).$$

The functions f_i being assumed to be analytical functions of their arguments, the transformation which corresponds to the parameters,

$$a_k^0 + c_k\,\delta t,$$

where δt is an infinitesimal and c_k an arbitrary constant, will convert x_i into x_i' where the difference

$$x_i' - x_i = \delta x_i$$

will be, in general, an infinitesimal of the same order as δt. We shall find, in fact

$$\delta x_i = \sum_{k=1}^{r}\left(\frac{\partial f_i}{\partial a_k}\right)_0 c_k\,\delta t + \cdots = \sum_{k=1}^{r} c_k\,\xi_{ki}\,\delta t + \cdots,$$

where $\left(\frac{\partial f_i}{\partial a_k}\right)_0 = \xi_{ki}(x_1, \ldots x_n)$ denotes the value of $\frac{\partial f_i}{\partial a_k}$ for $a_l = a_l^0$, and is therefore a function of $x_1, \ldots x_n$ only. The constants $c_1, \ldots c_r$ are arbitrary.

The transformations

$$(5) \qquad \delta x_i = \sum_{k=1}^{r} c_k\,\xi_{ki}(x_1, \ldots x_n)\,\delta t, \quad (i = 1, 2, \ldots n)$$

are called by *Lie*, the *infinitesimal transformations* of the group. In some cases they cannot be obtained in the way indicated. But we need not insist upon these exceptional cases, as we shall not need them in the course of this work.

If we consider an arbitrary function of $x_1, \dots x_n$, say

$$f(x_1, x_2, \dots x_n),$$

the infinitesimal change in f, which results from the infinitesimal transformation (5), is

$$\delta f = \left(\sum_{k=1}^{r} c_k U_k f\right) \delta t = Uf \cdot \delta t \tag{6}$$

where

$$U_k f = \xi_{k1} \frac{\partial f}{\partial x_1} + \xi_{k2} \frac{\partial f}{\partial x_2} + \dots + \xi_{kn} \frac{\partial f}{\partial x_n}. \tag{7}$$

Lie speaks of Uf as the *symbol* of the infinitesimal transformations (5) of the group. We have

$$Uf = c_1 U_1 f + c_2 U_2 f + \dots + c_r U_r f,$$

$U_1 f, \dots U_r f$ being themselves the symbols of infinitesimal transformations which are contained in the group In fact Uf reduces to $U_k f$ for

$$c_1 = c_2 = \dots = c_{k-1} = c_{k+1} = \dots = c_r = 0, \quad c_k = 1.$$

The r infinitesimal transformations, whose symbols are $U_1 f, \dots U_r f$, are said to be *linearly independent*, if it is impossible to find r non-vanishing constants, so that for an arbitrary function $f(x_1, \dots x_n)$, the equation

$$c_1 U_1 f + c_2 U_2 f + \dots + c_r U_r f = 0$$

will hold

We then have the following theorem: *An r-parameter continuous group contains precisely r linearly independent infinitesimal transformations*

From two expressions $U_1 f$ and $U_2 f$ of the form (7) we may form the *commutator*, (Klammerausdruck),

$$U_1(U_2 f) - U_2(U_1 f) = (U_1, U_2) f, \tag{8}$$

which is again of the same form. In fact, the second derivatives of f eliminate, and we find

$$(U_1, U_2) f = \sum_{i=1}^{n} [U_1(\xi_{2i}) - U_2(\xi_{1i})] \frac{\partial f}{\partial x_i}. \tag{9}$$

Lie has shown that r *infinitesimal transformations*

$$U_1 f, \dots U_r f$$

are precisely the r infinitesimal transformations of an r parameter group, if and only if they satisfy the relations

$$(U_i, U_k) f = \sum_{l=1}^{r} c_{ikl} U_l f, \tag{10}$$

where the quantities c_{ikl} are constants.

In this case $U_1 f, \ldots U_r f$ are said to *generate* an r-parameter continuous group.

If a function $f(x_1, \ldots x_n)$ remains unchanged by all of the transformations of the group, it is said to be an *invariant*. In particular, an invariant will not be changed by any *infinitesimal* transformation of the group; it must, therefore, satisfy the r partial differential equations

$$U_k f = 0, \quad (k = 1, 2, \ldots r). \tag{11}$$

Lie has shown that this *necessary* condition for invariants, is also *sufficient*, whenever the group may be generated by infinitesimal transformations. All invariants of the group may, therefore, be found by integrating a system of partial differential equations of the form (11).

But in regard to this system (11) we may make the following remarks. Although $U_1 f, \ldots U_r f$ were linearly independent as infinitesimal transformations, the r equations (11) need not be independent. For there may be relations of the form

$$\varphi_1(x_1, \ldots x_n)\, U_1 f + \cdots + \varphi_r(x_1, \ldots x_n)\, U_r f = 0,$$

where $\varphi_1, \ldots \varphi_r$ are functions of $x_1, \ldots x_n$, even though $U_1 f, \ldots U_r f$ be linearly independent in the former sense Suppose then, that q of the equations (11) are independent ($q < r$), and let these equations be

$$U_1 f = 0, \quad U_2 f = 0, \ldots U_q f = 0. \tag{12}$$

Let u be any solution of (12). Then clearly

$$U_i(U_k u) = 0, \quad U_k(U_i u) = 0,$$

whence

$$(U_i, U_k)\, u = 0,$$

i. e. any solution of (12) will also satisfy all of the further equations which can be obtained from (12) by the commutator operation. If the equations (12) are taken at random, we shall obtain in this way successively new equations which any solution of (12) must satisfy. We shall find finally a system of the form (12) such that all of the commutators formed from it will be zero as a consequence of the system itself. Such a system has been called a *complete system* by *Clebsch*.[1]) The general theory of complete systems is due to *Jacobi*.[2]) From our above considerations it follows that *the invariants of an r parameter group may be obtained by integrating a complete system of $q \leqq r$ partial differential equations of the first order in n independent variables.*

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1) *Clebsch*, Crelle's Journal, vol. 65.

2) *Jacobi*, ibid vol 60 Cf also the first volume of *Lie-Engel*, Transformationsgruppen

But according to the general theory, a complete system of q equations with n independent variables has precisely $n-q$ independent solutions, of which all other solutions are functions.

If, therefore, an r parameter group in n variables, is generated by the r infinitesimal transformations $U_1 f, \ldots U_r f$, and if among the r equations

$$U_k f = 0, \quad (k = 1, 2, \ldots r), \tag{13}$$

q are independent, the group has precisely $n-q$ invariants, which are obtained by integrating the complete system to which (13) *is equivalent.*

We have defined the term; an r-parameter group. But a system of transformations may have the group property although its equations cannot be expressed by a finite number of parameter. For example, the transformations

$$y' = \lambda(x) y, \quad x' = \mu(x) \tag{14}$$

where λ and μ are arbitrary functions of x, clearly have the group property; i. e. if we make successively two transformations of this kind, the result is another transformation of the same character.

Following *Lie* we shall, therefore, distinguish between *finite* and *infinite* continuous groups. The former contain only a finite number of arbitrary constants in their general equations, while the latter contain an infinite number of such constants, or arbitrary functions. The general theory of infinite groups has not been constructed. There exists, however, an important class of infinite groups for which a general invariant theory, (due to *Lie*), exists. Let

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \cdots + \xi_n \frac{\partial f}{\partial x_n}$$

be the most general infinitesimal transformation of the group. It may happen that there exists a system of linear homogeneous partial differential equations

$$a_{k1} \xi_1 + \cdots + a_{kn} \xi_n + b_{k1} \frac{\partial \xi_1}{\partial x_1} + \cdots = 0, \quad (k = 1, 2, \ldots n)$$

of which $\xi_1, \ldots \xi_n$ are solutions. In that case *the group is said to be defined by differential equations.* Such is the case in the above example. We have

$$Uf = \xi_1 \frac{\partial f}{\partial x} + \xi_2 \frac{\partial f}{\partial y}, \quad \xi_1 = \psi(x), \quad \xi_2 = \varphi(x) y,$$

where φ and ψ are arbitrary functions of x. The defining differential equations are

$$\frac{\partial \xi_1}{\partial y} = 0, \quad \frac{\partial^2 \xi_2}{\partial y^2} = 0.$$

Lie has shown[1]) that *the invariants of an infinite continuous group, which is defined by differential equations, may always be obtained as the solutions of a complete system of partial differential equations, as in the case of finite groups.*

This complete system will, of course, be obtained by equating to zero the symbol of the most general infinitesimal transformation of the group.

CHAPTER I.

TRANSFORMATION OF SYSTEMS OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS.

Let there be given a system of n independent linear, homogeneous differential equations between n unknown functions $y_1, y_2, \ldots y_n$ of x, and let the order of the highest derivative which is present in the system be m. We may assume that none of these equations are of the zero$^{\text{th}}$ order, i. e that no equation of the system has the form, (free from all derivatives),

$$\sum_{\lambda=1}^{n} p_\lambda y_\lambda = 0,$$

since, by means of such an equation, we could reduce the number n of unknown functions. We may, for the same reason, assume that no combination of the equations of the system give rise to an equation of this special form. The system of equations will be assumed to be written in the form

$$
(1)\quad
\begin{aligned}
&y_i^{(m)} \quad + \sum_{\lambda=1}^{n} (p_{m-1,i,\lambda}\, y_\lambda^{(m-1)} + \cdots + p_{1,i,\lambda}\, y'_\lambda + p_{0,i,\lambda}\, y_\lambda) = 0, \\
&\qquad\qquad (i = 1, 2, \ldots \lambda_1), \\
&y_j^{(m-1)} + \sum_{\lambda=1}^{n} (p_{m-2,j,\lambda}\, y_\lambda^{(m-2)} + \cdots + p_{1,j,\lambda}\, y'_\lambda + p_{0,j,\lambda}\, y_\lambda) = 0, \\
&\qquad\qquad (j = \lambda_1 + 1, \lambda_1 + 2, \ldots \lambda_1 + \lambda_2), \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&y'_\sigma \quad + \sum_{\lambda=1}^{n} p_{0\sigma\lambda}\, y_\lambda = 0, \\
&(\sigma = \lambda_1 + \lambda_2 + \cdots + \lambda_{m-1} + 1, \ldots \lambda_1 + \lambda_2 + \cdots + \lambda_{m-1} + \lambda_m),
\end{aligned}
$$

where, of course

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_m = n.$$

1) *Lie*, Math Annalen, vol 24

The integration of (1) would involve

$$(2) \qquad M = \lambda_1 m + \lambda_2 (m-1) + \lambda_3 (m-2) + \cdots + \lambda_m$$

arbitrary constants. In fact according to the fundamental theorem of the theory of linear differential equations[1]), if for $x = x_0$ the coefficients p_{ikl} can be developed in series proceeding according to positive integral powers of $x - x_0$, convergent for values of x which make

$$|x - x_0| < \varrho,$$

there exists a system of functions $y_1, \ldots y_n$, expressible by power series convergent in the same domain, for which the values of the M quantities;

$$(3) \qquad \begin{array}{l} y_1, \quad \ldots y_{\lambda_1};\ y_1', \ldots y_{\lambda_1}';\ \ldots;\ y_1^{(m-1)}, \ldots y_{\lambda_1}^{(m-1)}; \\ y_{\lambda_1+1}, \ldots y_{\lambda_1+\lambda_2};\ \ldots\ldots\ldots;\ y_{\lambda_1+1}^{(m-2)}, \ldots y_{\lambda_1+\lambda_2}^{(m-2)}; \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ y_{\lambda_1+\lambda_2+\cdots+\lambda_{m-1}+1}, \ldots\ldots\ldots\ldots\ldots y_{\lambda_1+\lambda_2+\cdots+\lambda_{m-1}+\lambda_m}; \end{array}$$

for $x = x_0$ are arbitrarily assigned constants, and which satisfy the system of differential equations.

If, for a moment, $y_1, y_2, \ldots y_n, x$ are interpreted as coordinates of a point in an $n+1$ dimensional space, we may speak of any transformation of the form

$$(4) \qquad y_i = g_i(\xi;\ \eta_1, \eta_2, \ldots \eta_n), \quad x = f(\xi;\ \eta_1, \eta_2, \ldots \eta_n),$$

where g_i and f are arbitrary functions of their arguments which do not contain the derivatives $\frac{d\eta_i}{d\xi}$, etc., as a *point-transformation*. For, if these functions are independent, i. e. if the Jacobian of $g_1, g_2, \ldots g_n, f$ with respect to $\eta_1, \ldots \eta_n, \xi$ is not identically zero, the point $(y_1, y_2, \ldots y_n, x)$ will then be transformed into another point $(\eta_1, \eta_2, \ldots \eta_n, \xi)$ of the $n+1$ dimensional space considered, and vice versa.

We shall assume that the functions f and g_i admit differentiation up to the order required by the following considerations. We may, although this restriction is not necessary, assume that they are analytic functions of their arguments.

A transformation of the form (4) applied to (1), will give rise to a new system of differential equations, obviously of the same order, but not in general linear and homogeneous. We wish to find

1) The proof, essentially upon function theoretic principles due to *Cauchy*, for the case of a single linear differential equation was given by *Fuchs*, Crelle's Journal vol. 66 It is a very easy matter to extend this proof to the case of a system of form (1)

the most general form of the transformation compatible with the condition that the transformed system may also be linear and homogeneous; we assume moreover that the coefficients of (1) may be any analytic functions of x, and that the functions f and g_i are not dependent upon these coefficients. In other words, the transformation (4) when found, is such as to convert every system whatever of form (1) into another of the same kind, and it is the same transformation for all systems of this form. If, for example, $\mathfrak{y}_1, \dots \mathfrak{y}_n$ are solutions of (1), the equations

$$y_k = \eta_k + \mathfrak{y}_k \quad (k = 1, 2, \dots n), \quad x = \xi,$$

would transform (1) into another system of the same form, in fact, into itself. But such transformations are excluded, because the solutions $\mathfrak{y}_k$ depend upon the coefficients of (1). This transformation is a different one for different systems of the form (1).

We find, from (4), by successive differentiation

$$\frac{dy_i}{dx} = \frac{Y_{i1}}{\sigma}, \quad \frac{d^2 y_i}{dx^2} = \frac{Y_{i2}}{\sigma^3}, \dots \frac{d^\mu y_i}{dx^\mu} = \frac{Y_{i\mu}}{\sigma^{2\mu-1}}, \tag{5}$$

where

$$\sigma = \frac{\partial f}{\partial \xi} + \sum_{\lambda=1}^{n} \frac{\partial f}{\partial \eta_\lambda} \frac{d\eta_\lambda}{d\xi},$$

(6)

$$Y_{i1} = \frac{\partial g_i}{\partial \xi} + \sum_{\lambda=1}^{n} \frac{\partial g_i}{\partial \eta_\lambda} \frac{d\eta_\lambda}{d\xi}, \quad (i = 1, 2, \dots n),$$

and where $Y_{i2}, Y_{i3}, \dots$ are defined by these equations as rational integral functions of η_λ', η_λ'', etc., if we denote derivatives with respect to ξ by strokes. We have, in particular

$$Y_{i2} = \sigma \frac{dY_{i1}}{d\xi} - Y_{i1} \frac{d\sigma}{d\xi}.$$

Let $H_{i2\lambda}$ be the coefficient of η_λ'' in this expression. Then

$$H_{i2\lambda} = \frac{\partial g_i}{\partial \eta_\lambda}\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta_\lambda}\frac{\partial g_i}{\partial \xi} + \sum_{\mu=1}^{n} \left(\frac{\partial g_i}{\partial \eta_\lambda}\frac{\partial f}{\partial \eta_\mu} - \frac{\partial f}{\partial \eta_\lambda}\frac{\partial g_i}{\partial \eta_\mu} \right) \eta_\mu', \tag{7}$$

$$(i, \lambda = 1, 2, \dots n).$$

For a fixed value of i, all of these quantities $H_{i2\lambda}$ cannot be equal to zero for all values of η_k and η_k'. For, if they were, all of the Jacobians

$$\frac{\partial g_i}{\partial \eta_\lambda}\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta_\lambda}\frac{\partial g_i}{\partial \xi}, \quad \frac{\partial g_i}{\partial \eta_\lambda}\frac{\partial f}{\partial \eta_\mu} - \frac{\partial f}{\partial \eta_\lambda}\frac{\partial g_i}{\partial \eta_\mu}$$

would be zero, i. e. the functions f and g_i would not be independent.

From the equation

$$\frac{d^{\mu-1}y}{dx^{\mu-1}} = \frac{Y_{i,\mu-1}}{\sigma^{2\mu-3}},$$

we obtain by differentiation

$$\frac{d^{\mu}y}{dx^{\mu}} = \frac{\sigma\frac{dY_{i,\mu-1}}{d\xi} - (2\mu-3)\frac{d\sigma}{d\xi}Y_{i,\mu-1}}{\sigma^{2\mu-1}},$$

so that

(8) $$Y_{i,\mu} = \sigma\frac{dY_{i,\mu-1}}{d\xi} - (2\mu-3)\frac{d\sigma}{d\xi}Y_{i,\mu-1}.$$

Let $H_{i\mu\lambda}$ denote the coefficient of $\eta_\lambda^{(\mu)}$ in $Y_{i\mu}$. Then (8) shows that

$$H_{i\mu\lambda} = \sigma H_{i,\mu-1,\lambda}, \quad (\mu = 3, 4, \ldots),$$

whence

(9) $$H_{i\mu\lambda} = \sigma^{\mu-2}H_{i2\lambda},$$

so that $H_{i\mu\lambda}$ is different from zero if $H_{i2\lambda}$ does not vanish; for

$$\sigma = \frac{dx}{d\xi}$$

cannot be zero.

Let us substitute (5) in (1). We find

(10) $$Y_{im} + \sum_{k=1}^{n}(p_{m-1,i,k}Y_{k,m-1}\sigma^2 + p_{m-2,i,k}Y_{k,m-2}\sigma^4 + \cdots$$
$$+ p_{1ik}Y_{k1}\sigma^{2m-2} + p_{0ik}g_k\sigma^{2m-1}) = 0,$$
etc. . . . etc.,

where we imagine the coefficients $p_{\lambda\mu\nu}$ also expressed in terms of $\xi, \eta_1, \ldots \eta_n$.

Y_{im} is linear in $\eta_1^{(m)}, \ldots \eta_n^{(m)}$, and actually contains at least one of these m^{th} derivatives, since at least one of the quantities $H_{i2\lambda}$ and, therefore, at least one of the quantities $H_{im\lambda}$ is different from zero.

It must be possible, if the transformation is of the required character, to solve (10) for λ_1 derivatives of order m, say $\eta_1^{(m)}, \ldots \eta_{\lambda_1}^{(m)}$; for λ_2 derivatives of order $m-1$, say $\eta_{\lambda_1+1}^{(m-1)}, \ldots \eta_{\lambda_1+\lambda_2}^{(m-1)}$; etc. We shall then have a new system of differential equations of the form (1), which we may imagine written down replacing the Roman throughout by the corresponding Greek letters. From this system, which we may denote by (1)$'$, certainly all of the m^{th} derivatives $\eta_\lambda^{(m)}$ can be expressed in the form

(11) $$\eta_\lambda^{(m)} = \sum_{k=1}^{n}(q_{m-1,\lambda,k}\eta_k^{(m-1)} + q_{m-2,\lambda,k}\eta_k^{(m-2)} + \cdots + q_{0,\lambda,k}\eta_k),$$

where $q_{\mu\nu\varrho}$ are functions of ξ alone.

We have, on the other hand

$$Y_{\iota m} = \sum_{l=1}^{n} H_{\iota m l}\, \eta_l^{(m)} + \cdots,$$

where the terms not written are of order lower than m, and where $H_{\iota m \lambda}$ is an integral rational function of η_l' of degree $m-1$.

By hypothesis (10), when solved, gives rise to (1)', from which follows (11). The left members of (10) contain the highest derivatives only in the combinations

$$\Sigma H_{\iota m \lambda}\, \eta_l^{(m)}\ ^{1)}$$

These left members must, therefore, be obtained from (11) by making precisely these linear combinations. The first equation of (10) can, therefore, contain $\eta_1', \ldots \eta_n'$ to the m^{th} degree only, since $H_{\iota m l}$ is of degree $m-1$ in these quantities. But this equation contains the term

$$p_{0 \iota \lambda}\, g_\lambda\, \sigma^{2m-1}$$

which is of degree $2m-1$ in $\eta_1', \ldots \eta_n'$, provided that f depends upon any of the quantities $\eta_1, \ldots \eta_n$. For $m > 1$ this is a contradiction. Therefore, if $m > 1$, we must have f independent of $\eta_1, \ldots \eta_n$, i. e.

$$\frac{\partial f}{\partial \eta_l} = 0, \quad x = f(\xi).$$

We have, therefore,

$$\sigma = f'(\xi), \quad H_{\iota 2 \lambda} = f'(\xi) \frac{\partial g_\iota}{\partial \eta_l}, \quad H_{\iota n l} = f'(\xi)^{n-1} \frac{\partial g_\iota}{\partial \eta_l},$$

so that the coefficients of the highest derivatives in (10) are now free from the derivatives η_λ'. Each of the terms of (10) must therefore be linear in η_λ'. One of these terms is

$$p_{2 \iota \lambda}\, Y_{\lambda 2}\, \sigma^{2m-4}.$$

But we find

$$Y_{\lambda 2} = \sigma \frac{dY_{\lambda 1}}{d\xi} - Y_{\lambda 1} \frac{d\sigma}{d\xi},$$

where

$$\frac{dY_{\lambda 1}}{d\xi} = \frac{\partial^2 g_\iota}{\partial \xi^2} + 2 \sum_{l=1}^{n} \frac{\partial^2 g_\iota}{\partial \xi\, \partial \eta_l} \eta_l' + \sum_{l=1}^{n} \sum_{\mu=1}^{n} \frac{\partial^2 g_\iota}{\partial \eta_l\, \partial \eta_\mu} \eta_l' \eta_\mu' + \sum_{l=1}^{n} \frac{\partial g_\iota}{\partial \eta_l} \eta_l'',$$

so that $Y_{\lambda 2}$ is linear in η_λ' only if

---

1) To be sure all of the m^{th} derivatives, except $\eta_1^{(m)}, \ldots \eta_{\lambda_1}^{(m)}$, may be removed by means of the other equations of the system (1)'. But this does not affect the conclusion. More symmetrically, system (1) could be replaced by a system of n equations each of the m^{th} order, a system of λ_2 linear relations between the derivatives up to the $m-1^{\text{th}}$ order, a system of λ_3 relations between the derivatives up to the $m-2^{\text{th}}$ order, etc. System (1)' could be written in the same way and thus the conclusion would be rendered more obvious.

$$\frac{\partial^2 g_\iota}{\partial \eta_\lambda \partial \eta_\mu} = 0,$$

i. e. if

$$g_k = \alpha_{k1}(\xi)\eta_1 + \alpha_{k2}(\xi)\eta_2 + \cdots + \alpha_{kn}(\xi)\eta_n + \alpha_{k0}(\xi).$$

But in this case

$$H_{\iota\mu\lambda} = f'(\xi)^{n-1}\alpha_{\iota\prime}(\xi),$$

i. e. the coefficients of the highest derivatives in (10) are mere functions of ξ. Every term of (10) must therefore be linear and *homogeneous* in $\eta_1, \ldots \eta_n$ and the derivatives of these quantities. The term

$$p_{0\iota k}\, g_k\, \sigma^{2m-1}$$

is homogeneous, only if

$$\alpha_{k0}(\xi)$$

is zero.

If $m > 1$ the most general transformation of the kind required is, therefore,

$$x = f(\xi), \quad y_k = \alpha_{k1}(\xi)\eta_1 + \alpha_{k2}(\xi)\eta_2 + \cdots + \alpha_{kn}(\xi)\eta_n,$$
$$(k = 1, 2, \ldots n)$$

where $f(\xi)$ and $\alpha_{k\iota}(\xi)$ are arbitrary functions of ξ and where the determinant

$$\alpha_{\iota k}$$

is different from zero.

We still have to examine the case $m = 1$. Let

(12) $$\frac{dy_k}{dx} = p_{k1}y_1 + \cdots + p_{kn}y_n, \quad (k = 1, 2, \ldots n),$$

be the given system, and suppose that the transformation

$$x = f(\xi;\ \eta_1, \ldots \eta_n), \quad y_k = g_k(\xi;\ \eta_1, \ldots \eta_n)$$

converts it into

(12a) $$\frac{d\eta_k}{d\xi} = \pi_{k1}\eta_1 + \cdots + \pi_{kn}\eta_n, \quad (k = 1, 2, \ldots n).$$

We may write (12) in a different form. If we differentiate each equation $n - 1$ times, we shall find

$$\frac{d^\lambda y_k}{dx^\lambda} = p_{k\lambda 1}y_1 + \cdots + p_{k\lambda n}y_n, \quad (k, \lambda = 1, 2, \ldots n).$$

Eliminating the $n - 1$ quantities y_ι, $(i \neq k)$, from these equations, we shall find

(13) $$r_{kn}\frac{d^n y_k}{dx^n} + r_{k,n-1}\frac{d^{n-1}y_k}{dx^{n-1}} + \cdots + r_{k0}y_k = 0,$$
$$(k = 1, 2, \ldots n);$$

in special cases some of these equations may be of lower than the n^{th} order, but in general they are not.

The equations (13) are all integrated if one of them is. We shall have, in fact

$$
(14)\quad
\begin{aligned}
y_k &= s_{k0}\, y_1 + s_{k1}\frac{dy_1}{dx} + \cdots + s_{k,n-1}\frac{d^{n-1}y_1}{dx^{n-1}}, \quad (k = 2, 3, \ldots n),\\
y_k &= t_{k0}\, y_2 + t_{k1}\frac{dy_2}{dx} + \cdots + t_{k,n-1}\frac{d^{n-1}y_2}{dx^{n-2}}, \quad (k = 1, 3, \ldots n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_k &= v_{k0}\, y_n + v_{k1}\frac{dy_n}{dx} + \cdots + v_{k,n-1}\frac{d^{n-1}y_n}{dx^{n}}, \quad (k = 1, 2, \ldots n-1),
\end{aligned}
$$

where the coefficients $s_{ki}, t_{ki}, \ldots v_{ki}$ as well as the coefficients r_{ki} of of (13) may be obtained from p_{ik} by rational processes including differentiation. The combined systems (13) and (14) are equivalent to (12), the equations (14) serving to reduce the order of (13), which would be n^2, to n. In place of (12a) we may consider a corresponding combined system (13a) and (14a). The required transformation must convert (13) and (14) into a system equivalent to (13a) and (14a). By the method of the general case we find the same result as before, provided that $n > 1$.

There remains the case $m = n = 1$, so that the differential equation has the form

$$\frac{dy}{dx} + py = 0.$$

It is convenient in this case to write

$$\log y = y_1, \quad \log \eta = \eta_1.$$

We must then determine those transformations

$$(15)\qquad x = f_1(\xi, \eta_1), \quad y_1 = g_1(\xi, \eta_1)$$

which transform any equation of the form

$$(16)\qquad \frac{dy_1}{dx} + p(x) = 0,$$

into

$$(16\text{a})\qquad \frac{d\eta_1}{d\xi} + \pi(\xi) = 0.$$

We find, by applying directly the transformation (15) to (16),

$$(17)\qquad \frac{d\eta_1}{d\xi} + \frac{p\dfrac{\partial f_1}{\partial \xi} + \dfrac{\partial g_1}{\partial \xi}}{p\dfrac{\partial f_1}{\partial \eta_1} + \dfrac{\partial g_1}{\partial \eta_1}} = 0.$$

The second term of this equation must be a function of ξ only. The partial derivative with respect to η_1 must therefore be zero, for all possible functions p. Therefore, the equation

$$\left(\frac{dp}{dx}\frac{\partial f_1}{\partial \eta_1}\frac{\partial f_1}{\partial \xi}+p\frac{\partial^2 f_1}{\partial \xi\,\partial \eta_1}+\frac{\partial^2 g_1}{\partial \xi\,\partial \eta_1}\right)\left(p\frac{\partial f_1}{\partial \eta_1}+\frac{\partial g_1}{\partial \eta_1}\right)$$
$$-\left[\frac{dp}{dx}\left(\frac{\partial f_1}{\partial \eta_1}\right)^2+p_1\frac{\partial^2 f_1}{\partial \eta_1^2}+\frac{\partial^2 g_1}{\partial \eta_1^2}\right]\left(p\frac{\partial f_1}{\partial \xi}+\frac{\partial g_1}{\partial \xi}\right)=0,$$

must be satisfied identically. This is impossible unless the coefficient of $\frac{dp}{dx}$ is zero, i. e. unless

$$\frac{\partial f_1}{\partial \eta_1}\left(\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta_1}-\frac{\partial f_1}{\partial \eta_1}\frac{\partial g_1}{\partial \xi}\right)=0.$$

The second factor cannot be zero, since f_1 and g_1 are independent functions of ξ and η_1. Therefore

$$\frac{\partial f_1}{\partial \eta_1}=0,$$

i. e. f_1 is a function of ξ only, so that instead of (15) we may write

$$x=f_1(\xi),\quad y_1=g_1(\xi,\eta_1).$$

We find in this case, in place of (17),

$$\frac{d\eta_1}{d\xi}+\frac{p(f_1)f_1'(\xi)+\frac{\partial g_1}{\partial \xi}}{\frac{\partial g_1}{\partial \eta_1}}=0,$$

where again the second term must be a function of ξ alone. Since $p(x)$ was an arbitrary function of x, we see first that $\frac{\partial g_1}{\partial \xi}:\frac{\partial g_1}{\partial \eta_1}$ must be a function of ξ only, since the second term reduces to this ratio for $p=0$. But if $p \neq 0$, we see in the second place that $\frac{\partial g_1}{\partial \eta_1}$ must be a mere function of ξ, and finally also $\frac{\partial g_1}{\partial \xi}$. g_1 is therefore linear in η_1, say

$$g_1=g_1(\xi)+g_2(\xi)\eta,$$

whence

$$\frac{\partial g_1}{\partial \xi}=g_1'(\xi)+g_2'(\xi)\eta.$$

But this latter expression must be a function of ξ only, so that

$$g_2=\lambda$$

where λ is a constant We have therefore the following transformation

$$x=f_1(\xi), y_1=\lambda\eta_1+g_1(\xi)$$

whence

(18) $$x=f(\xi),\quad y=g(\xi)\eta^\lambda,\quad \lambda=const.$$

Therefore (18) is the most general transformation which can convert a general homogeneous linear differential equation of the first order into another of the same kind. It is easy to verify that every transformation of this form actually accomplishes this.

We may recapitulate our result as follows. *The most general point-transformation, which converts a system of n linear differential equations into another of the same form and order, is*

$$\xi = f(x), \quad \eta_k = \alpha_{k1}(x)y_1 + \alpha_{k2}(x)y_2 + \cdots + \alpha_{kn}(x)y_n,$$
$$(k = 1, 2, \ldots n),$$

where $f(x)$ and $\alpha_{ki}(x)$ are arbitrary functions of x, for which the determinant

$$|\alpha_{ki}(x)|, \quad (i, k = 1, 2, \ldots n),$$

does not vanish.

If $n = 1$, and if the single differential equation, to which the system then reduces, is of the first order there is an exception. In that case the most general transformation, which has the required property, is

$$x = f(\xi), \quad y = g(\xi)\eta^\lambda,$$

where f and g are arbitrary functions, and λ an arbitrary constant.

For the case of a single linear differential equation, the transformation becomes, $(m > 1)$,

$$\xi = f(x), \quad \eta = g(x)y.$$

The proof, that this is the most general transformation converting every linear differential equation of the m^{th} order into another, was first given by *Stäckel*.[1]) The generalization to systems of differential equations is due to the author[2]) A shorter, but less elementary proof than that of *Stäckel* is due to *Lie*[3])

CHAPTER II.

INVARIANTS OF THE LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION OF THE n^{th} ORDER

§ 1. Fundamental Notions.

Let us consider the linear differential equation

$$(1) \qquad y^{(n)} + \binom{n}{1}p_1 y^{(n-1)} + \binom{n}{2}p_2 y^{(n-2)} + \cdots + p_n y = 0,$$

where the symbol

1) *Stäckel*, Crelle's Journal, vol 111 (1893), p. 290. *Stäckel* there also gives the investigation for $m = 1$ which we have reproduced

2) *Wilczynski*, Am. Journ of Math, vol. 23 (1901), p. 29.

3) *Lie*, Leipziger Berichte (1894), p. 322. *Lie* emphasizes the fact that such results are mere corollaries of his general theory. For the present work, however, they are of especial importance.

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

represents the coefficient of x^k in the expansion of $(1+x)^n$, where

$$y^{(i)} = \frac{d^i y}{dx^i},$$

and where $p_1, p_2, \dots p_n$ are functions of x.

We have seen that the most general transformation, which converts (1) into another equation of the same form and order, is

$$(2) \qquad x = f(\xi), \quad y = \lambda(\xi)\eta,$$

where $f(\xi)$ and $\lambda(\xi)$ are arbitrary functions of ξ. Clearly all of the transformations of the form (2) form a group, an infinite continuous group in the sense of *Lie*, which is defined by differential equations.

If the general transformation (2) be applied to (1) another equation of the same form will be obtained between η and ξ, whose coefficients will be expressible in terms of $p_1, \dots p_n$, f, λ, and of certain derivatives of these functions. We shall always suppose that all of these derivatives exist In fact we may assume for our purposes, although this involves an unnecessary restriction, that all of these functions are analytical. Any differential equation, which may be obtained from (1) by a transformation (2), shall be said to be *equivalent* to (1). A function of the coefficients $p_1, p_2, \dots p_n$ of (1) and of their derivatives, which has the same value as the same function formed for an equivalent equation, shall be called an *absolute invariant.* If such an invariant function also contains y, y', y'', etc., we shall speak of it as a *covariant.*

§ 2. Seminvariants and semi-covariants.

The transformation (2) may be conveniently decomposed into two others. Let us put first

$$(3) \qquad y = \lambda(x)\eta$$

where $\lambda(x)$ is an arbitrary function of x. This gives rise to a differential equation between η and x. We may then transform the independent variable by putting

$$x = f(\xi).$$

The transformation (3) clearly form a sub-group of (2) which is still, an *infinite* continuous group. We shall speak of the functions, which remain invariant under the transformations of this sub-group, as *seminvariants* and *semi-covariants*, and we shall proceed to determine them immediately. Since invariants must also be seminvariants we shall then be able to determine the invariants as special seminvariants, namely such as remain unchanged by an arbitrary transformation of the independent variable as well.

We find, from (3),

$$
(4)\qquad
\begin{aligned}
y &= \lambda\eta,\\
y' &= \lambda\eta' + \lambda'\eta,\\
&\cdots\cdots\\
y^{(n)} &= \lambda\eta^{(n)} + \binom{n}{1}\lambda'\eta^{(n-1)} + \binom{n}{2}\lambda''\eta^{(n-2)} + \quad + \lambda^{(n)}\eta,
\end{aligned}
$$

so that (1) becomes

$$
(5)\qquad \eta^{(n)} + \binom{n}{1}\pi_1\eta^{(n-1)} + \binom{n}{2}\pi_2\eta^{(n-2)} + \quad + \pi_n\eta = 0,
$$

where

$$
(6)\qquad
\begin{aligned}
\pi_1 &= \frac{1}{\lambda}\left[\lambda' + p_1\lambda\right],\\
\pi_2 &= \frac{1}{\lambda}\left[\lambda'' + 2p_1\lambda' + p_2\lambda\right],\\
&\cdots\cdots\\
\pi_n &= \frac{1}{\lambda}\left[\lambda^{(n)} + \binom{n}{1}p_1\lambda^{(n-1)} + \binom{n}{2}p_2\lambda^{(n-2)} + \quad + p_n\lambda\right],
\end{aligned}
$$

as may be found by direct computation. Without computation these equations may be found by noting that (4) is a linear homogeneous substitution in $n+1$ variables $y, y', \ldots y^{(n)}$, that the quantities

$$
z^{(n)} = p_0 = 1,\ z^{(n-1)} = \binom{n}{1}p_1,\ z^{(n-2)} = \binom{n}{2}p_2, \ldots z' = \binom{n}{n-1}p_{n-1},\ z = p_n
$$

constitute a second set of $n+1$ variables, and that the transformation of the latter set must be contragredient to that of the first, so as to leave the bilinear form (the differential equation)

$$
y^{(n)}z^{(n)} + y^{(n-1)}z^{(n-1)} + \quad + y'z' + yz = 0,
$$

invariant.

As equations (6) show, we may always choose $\lambda(x)$ so as to make π_1 vanish. We need only put

$$
\lambda = e^{-\int p_1 dx},
$$

so that (5) becomes

$$
(7)\qquad \eta^{(n)} + \binom{n}{2}P_2\eta^{(n-2)} + \binom{n}{3}P_3\eta^{(n-3)} + \cdots + P_n\eta = 0,
$$

where

$$
(8)\qquad P_k = e^{\int p_1 dx}\sum_{i=0}^{k}\binom{k}{i}p_i\frac{d^{k-i}\left(e^{-\int p_1 dx}\right)}{dx^{k-i}}, \quad (k = 2, 3, 4, \ldots n).
$$

We find in particular

$$
(9)\qquad
\begin{aligned}
&P_2 = p_2 - p_1^2 - p_1', \quad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'',\\
&P_4 = p_4 - 4p_1p_3 + 6p_1^2p_2 - 6p_1'p_2 - 3p_1^4 + 6p_1^2p_1'\\
&\qquad + 3p_1'^2 - p_1^{(3)}, \text{ etc.,}
\end{aligned}
$$

and it is evident that the exponentials of equation (8) always cancel each other. We may write, moreover,

$$(9a)\quad P_4 = p_4 - 4p_1p_3 + 12p_1^2p_2 - 3p_2^2 - 6p_1^4 - p_1^{(3)} + 3P_2^2,$$

a formula which we shall need later.

We have seen that any linear differential equation of the n^{th} order may be reduced to the form (7) in which the $n-1^{\text{th}}$ derivative is absent, and which shall be called its *semi-canonical form*. As long as the independent variable x is not transformed, the semi-canonical form is unique. For, although we might more generally have put

$$\lambda = Ce^{-\int p_1 dx},$$

where C is an arbitrary constant, this would not affect the coefficients P_k of the semi-canonical form of the equation.

Consider now any equation (5) which can be obtained from (1) by a transformation of the form $y = \lambda\eta$. If we reduce it to *its* semi-canonical form, the coefficients $\Pi_2, \Pi_3, \ldots \Pi_n$ of the latter will of course be precisely the same functions of $\pi_1, \pi_2, \ldots \pi_n$ as $P_2, \ldots P_n$ are of $p_1, p_2, \ldots p_n$. But, since (1) and (5) can be transformed into each other, their semi-canonical forms must coincide. Therefore

$$\Pi_k = P_k, \quad (k = 2, 3, \ldots n);$$

in other words, *the* $n-1$ *quantities* $P_2, P_3, \ldots P_n$ *are seminvariants.* Obviously the same is true of their derivatives of any order, and of any function of these quantities.

But the converse is also true, i. e. *every seminvariant is a function of* $P_2, P_3, \ldots P_n$ *and of the derivatives of these quantities.*

For, let

$$f(p_1, p_2, \ldots p_n;\ p_1', p_2', \ldots p_n', \text{ etc.})$$

be a seminvariant. It must be equal to the same function of the coefficients of any differential equation obtained from (1) by a transformation of the form (3). In particular it must therefore be equal to

$$f(0, P_2, \ldots P_n;\ 0, P_2', \ldots P_n'; \ldots),$$

where $P_2, \ldots P_n$ are the coefficients of the semi-canonical form, i. e. it must be a function of these quantities and of their derivatives, as we asserted.

Having found all of the seminvariants we proceed to determine the semi-covariants. We may confine our attention to semi-covariants which contain no higher derivatives of y than the $n-1^{\text{th}}$. For, if a semi-covariant contains $y^{(n)}, y^{(n+1)}$, etc. we may express these higher derivatives in terms of the lower ones, by means of the differential equation itself and of others derived from it by differentiation.

Put

$$(10)\quad y_k = y^{(k)} + \binom{k}{1} p_1 y^{(k-1)} + \binom{k}{2} p_2 y^{(k-2)} + \cdots + p_k y, \quad (k = 1, 2, \ldots n),$$

and denote by η_k the corresponding expression in η and π. Then it is clear without any computation that

$$(11)\qquad y_k = \lambda \eta_k.$$

For, equations (4) and (6) show that, just as the equation $y_n = 0$ is transformed into $\lambda \eta_n = 0$, so will y_k be transformed into $\lambda \eta_k$.

We have therefore, the following $n-1$ semi-covariants

$$(12)\qquad \frac{y_1}{y}, \frac{y_2}{y}, \ldots \frac{y_{n-1}}{y}.$$

Any semi-covariant must be a function of these and of seminvariants. For, in the first place, every absolute semi-covariant must be homogeneous of degree zero in $y, y', \ldots y^{(n-1)}$, since if we take $\lambda = const.$, equations (4) and (6) reduce to

$$(13)\qquad y^{(k)} = \lambda \eta^{(k)}, \quad p_k = \pi_k.$$

Any absolute semi-covariant must, therefore, be a function of

$$\frac{y'}{y}, \frac{y''}{y}, \ldots \frac{y^{(n-1)}}{y}; \quad p_1, p_2, \ldots p_n; \quad p_1', p_2', \ldots p_n'; \text{ etc}$$

By means of (10) this becomes a function of

$$\frac{y_1}{y}, \ldots \frac{y_{n-1}}{y}; \quad p_1, p_2, \ldots p_n; \text{ etc.}$$

But if it is a semi-covariant it must be equal to the corresponding function for the semi-canonical form, and must therefore reduce to a function of

$$\frac{y_1}{y}, \ldots \frac{y_{n-1}}{y}; \quad P_2, \ldots P_n; \quad P_2', \ldots P_n'; \text{ etc.}$$

This justifies the above statement

§ 3. Invariants and covariants. Fundamental properties.

Before proceeding to the explicit calculation of the invariants and covariants, it will be useful to deduce a few simple theorems about them

We have seen that any absolute semi-covariant must be homogeneous of degree zero in $y, y', \ldots y^{(n-1)}$. The same must therefore be true of an absolute covariant.

An *irreducible* rational integral expression

$$f(y, \ldots y^{(n-1)}; p_1, \ldots p_n; \ldots),$$

i. e. one which cannot be resolved into integral rational factors, is said to be a relative covariant, if the equation $f=0$ has as its consequence the same equation in the new variables, i. e.

$$f(\eta, \ldots \eta^{(n-1)}; \pi_1, \ldots \pi_n; \ldots) = 0.$$

Equations (13) show that *every covariant must be homogeneous in* $y, y', \ldots y^{(n-1)}$.

We proceed to make a very simple transformation of the form (2) by putting

$$y = \eta, \quad \xi = cx,$$

where c is a constant. We find

$$\frac{d^{\lambda}\eta}{d\xi^{\lambda}} = c^{-\lambda}\frac{d^{\lambda}y}{dx^{\lambda}}, \quad \frac{d^{l}\pi_r}{d\xi^{l}} = c^{-(r+l)}\frac{d^{l}p_r}{dx^{l}}.$$

If we assign to $[y^{(\lambda)}]^m$ the weight λm, and to $[p_r^{(l)}]^m$ the weight $(r+l)m$, we see that every term is multiplied by a power of c whose index is its weight. We see, therefore, that *every covariant must be isobaric, every absolute covariant isobaric of weight zero* Besides, as we have seen, it must be homogeneous in $y, y', \ldots y^{(n-1)}$, and of degree zero if it is an absolute covariant. Invariants are, of course, included among the covariants as special cases, their degree being zero

Let $\Omega^{(\lambda, w)}$ be a rational integral covariant of degree λ in $y, \ldots y^{(n-1)}$, and of weight w. Let us make the transformation (3). Equations (4) and (6) show that

$$\Omega^{(\lambda, w)} = \lambda^{k}\bar{\Omega}^{(\lambda, w)} + \Theta,$$

where $\bar{\Omega}^{(\lambda, w)}$ denotes the same function of the new variables as $\Omega^{(\lambda, w)}$ of the old, and where Θ is a rational function of lower weight than w. But the equation $\bar{\Omega}^{(\lambda, w)} = 0$ must be a consequence of $\Omega^{(k, w)} = 0$, which requires that Θ shall vanish. If we assume further that $\Omega^{(\lambda, w)}$ is irreducible, Θ cannot vanish as a consequence of $\Omega^{(\lambda, w)} = 0$, but must be identically zero. *An irreducible rational integral covariant, of degree* λ, *is therefore transformed in accordance with the equation*

$$\Omega^{(\lambda, w)} = \lambda^{k}\bar{\Omega}^{(k, w)},$$

if the dependent variable is transformed by (3).

Let the independent variable be transformed by putting

$$\xi = \xi(x).$$

We find

$$
(14)\qquad
\begin{aligned}
y' &= \xi' \frac{dy}{d\xi},\\
y'' &= (\xi')^2 \frac{d^2 y}{d\xi^2} + \xi'' \frac{dy}{d\xi},\\
y^{(3)} &= (\xi')^3 \frac{d^3 y}{d\xi^3} + 3\xi'\xi'' \frac{d^2 y}{d\xi^2} + \xi^{(3)} \frac{dy}{d\xi},\\
y^{(4)} &= (\xi')^4 \frac{d^4 y}{d\xi^4} + 6(\xi')^2\xi'' \frac{d^3 y}{d\xi^3} + [4\xi'\xi^{(3)} + 3(\xi'')^2] \frac{d^2 y}{d\xi^2} + \xi^{(4)} \frac{dy}{d\xi};
\end{aligned}
$$

in general

$$
(15)\qquad y^{(m)} = \sum_{\lambda=1}^{m} \frac{A_{m\lambda}}{\lambda!} \frac{d^\lambda y}{d\xi^\lambda}, \quad (m = 1, 2, 3, \ldots),
$$

where $A_{m\lambda}$ is an integral rational function of the derivatives of $\xi(x)$. Some of these coefficients we shall have to determine. We find at once from (14):

$$
(16)\qquad
\begin{aligned}
&A_{11} = \xi', \quad A_{21} = \xi'', \quad A_{22} = 2(\xi')^2;\\
&A_{31} = \xi''', \quad A_{32} = 6\xi'\xi'', \quad A_{33} = 6(\xi')^3;\\
&A_{41} = \xi^{(4)}, \quad A_{42} = 8\xi'\xi^{(3)} + 6(\xi'')^2, \quad A_{43} = 36(\xi')^2\xi'', \quad A_{44} = 24(\xi')^4.
\end{aligned}
$$

Differentiation of (15) gives

$$
y^{(m+1)} = \frac{A_{m,m}}{m!}\xi' \frac{d^{m+1} y}{d\xi^{m+1}} + \sum_{\lambda=2}^{m} \frac{A'_{m,\lambda} + \lambda\xi' A_{m,\lambda-1}}{\lambda!} \frac{d^\lambda y}{d\xi^\lambda} + A'_{m,1} \frac{dy}{d\xi},
$$

whence

$$
(17)\qquad
\begin{aligned}
A_{m+1,m+1} &= (m+1) A_{m,m}\xi',\\
A_{m+1,m} &= A'_{m,m} + m\xi' A_{m,m-1},\\
A_{m+1,m-1} &= A'_{m,m-1} + (m-1)\xi' A_{m,m-2},\\
&\cdots\cdots\cdots\cdots\\
A_{m+1,1} &= A'_{m,1}
\end{aligned}
$$

From the first and last of these equations, together with $A_{11} = \xi'$, we deduce at once

$$
(18)\qquad A_{m,m} = m!(\xi')^m, \quad A_{m,1} = \xi^{(m)}.
$$

From the second equation of (17) we find:

$$
\begin{aligned}
A_{m,m-1} &= A'_{m-1,m-1} + (m-1)\xi' A_{m-1,m-2},\\
A_{m-1,m-2} &= A'_{m-2,m-2} + (m-2)\xi' A_{m-2,m-3},\\
&\cdots\cdots\cdots\cdots\\
A_{3,2} &= A'_{2,2} + 2\xi' A_{2,1}.
\end{aligned}
$$

Multiply the members of these equations successively by

$$1,\ (m-1)\xi',\ (m-1)(m-2)(\xi')^2, \ldots (m-1)(m-2)\ldots 4\cdot 3(\xi')^{m-3}$$

and add. Then substitute the values of $A'_{m-1,m-1}$, etc. derived from (18), and the value of A_{21}. The result is

$$\frac{A_{m,m-1}}{(m-1)!} = \binom{m}{2}\xi''(\xi')^{m-2}. \tag{19}$$

From the third equation of (17) we find

$$\begin{aligned}
A_{m,m-2} &= A'_{m-1,m-2} + (m-2)\xi' A_{m-1,m-3},\\
A_{m-1,m-3} &= A'_{m-2,m-3} + (m-3)\xi' A_{m-2,m-4},\\
&\cdots\cdots\\
A_{4,2} &= A'_{3,2} + 2\xi' A_{3,1}.
\end{aligned}$$

Multiply the members of these equations in order, by

$$1,\ (m-2)\xi',\ (m-2)(m-3)(\xi')^2, \ldots, (m-2)(m-3)\ldots 3(\xi')^{m-4},$$

add, insert the values of $A'_{m-1,m-2}$, etc. from (19) and of $A_{3,1}$ from (16). The result is

$$\frac{A_{m,m-2}}{(m-2)!} = \binom{m}{3}\xi^{(3)}(\xi')^{m-3} + 3\binom{m}{4}(\xi'')^2(\xi')^{m-4}, \tag{20}$$

an equation for whose demonstration the following well-known formulae

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4},$$

may be used.

The substitution of (15) in (1) gives the result of the transformation $\xi = \xi(x)$. Denote the coefficients of the resulting equation by $\bar{p}_1, \ldots \bar{p}_n$. Then

$$\binom{n}{r}(\xi')^n \bar{p}_r = \sum_{i=0}^{r}\binom{n}{i}\frac{A_{n-i,n-r}}{(n-r)!}p_i, \quad (r = 0, 1, 2, \ldots n), \tag{21}$$

$$\bar{p}_0 = p_0 = 1.$$

According to (18) this shows that

$$\bar{p}_r = \frac{1}{(\xi')^r}p_r + q_{r-1} \tag{21a}$$

where q_{r-1} contains only terms of weight lower than r.

Solving (14), and denoting $\frac{\xi''}{\xi'}$ by η, we find

$$
\begin{aligned}
&\frac{dy}{d\xi} = \frac{1}{\xi'} y', \\
&\frac{d^2y}{d\xi^2} = \frac{1}{(\xi')^2} (y'' - \eta y'), \\
(22) \qquad &\frac{d^3y}{d\xi^3} = \frac{1}{(\xi')^3} [y^{(3)} - 3\eta y'' - (\eta' - 2\eta^2) y'], \\
&\frac{d^4y}{d\xi^4} = \frac{1}{(\xi')^4} [y^{(4)} - 6\eta y^{(3)} - (4\eta' - 11\eta^2) y'' - (\eta'' - 7\eta\eta' + 6\eta^3) y], \\
&\quad . \quad . \quad . \quad . \qquad\qquad\qquad . \quad . \quad . \quad . \quad .
\end{aligned}
$$

so that in general

$$
(22\text{a}) \qquad \frac{d^r y}{d\xi^r} = \frac{1}{(\xi')^r} y^{(r)} + Y_{r-1},
$$

where Y_{r-1} contains terms of weight no higher than $r-1$. From (21a) finally, one finds

$$
(21\text{b}) \qquad \frac{d^k \bar{p}_r}{d\xi^k} = \frac{1}{(\xi')^{r+k}} p_r^{(k)} + q_{r+k-1},
$$

where q_{r+k-1} is of weight no higher than $r+k-1$.

Let $\Omega^{(k,w)}$ be an integral, rational, irreducible covariant, homogeneous of degree k and isobaric of weight w. The equations (22a) and (21b) show that the transformation $\xi = \xi(x)$ will convert it into

$$
\bar{\Omega}^{(k,w)} = (\xi')^{-w} \Omega^{(k,w)} + \text{terms of lower weight.}
$$

But the equation $\bar{\Omega}^{(k,w)} = 0$ must follow from $\Omega^{(k,w)} = 0$ The terms of lower weight cannot vanish in consequence of the latter equation, since $\Omega^{(k,w)}$ is irreducible. They must therefore vanish identically. We have therefore the following theorem.

If $\Omega^{(k,w)}$ is an integral, rational, irreducible covariant of degree k and of weight w, the transformation

$$
(T) \qquad \bar{x} = \xi(x), \quad \bar{y} = \lambda(x) y,
$$

transforms it in accordance with the equation

$$
(23) \qquad \bar{\Omega}^{(k,w)} = \frac{\lambda^k}{(\xi')^w} \Omega^{(k,w)}.
$$

When three such covariants are known, an absolute covariant can always be constructed.

In fact, let $\Omega^{(k_i,w_i)}$ for $(i = 1, 2, 3)$ be three such covariants. Then

$$
[\Omega^{(k_1,w_1)}]^{\mu_1} [\Omega^{(k_2,w_2)}]^{\mu_2} [\Omega^{(k_3,w_3)}]^{\mu_3}
$$

is an absolute covariant, if

$$k_1\mu_1 + k_2\mu_2 + k_3\mu_3 = 0,$$
$$w_1\mu_1 + w_2\mu_2 + w_3\mu_3 = 0.$$

From these equations $\mu_1 : \mu_2 : \mu_3$ may be determined, except if

$$k_1 : k_2 : k_3 = w_1 : w_2 : w_3,$$

in which case two of the covariants suffice to determine an absolute covariant. This takes place, in particular, if two of the covariants are invariants.

Let U and V be two integral rational functions of the quantities $p_k^{(i)}$ and $y^{(r)}$, without a common factor. Let their quotient $I = \frac{U}{V}$ be an absolute invariant, and let c be a constant. Then the equation $I = c$ is an invariant equation, which, since U and V have no common factor, may be written

$$U - cV = 0.$$

But this equation, being an invariant integral rational equation, must be homogeneous in the y's say of degree k, and isobaric, say of weight w. Therefore

$$\bar{U} - c\bar{V} = \frac{\lambda^k}{(\xi')^w}(U - cV).$$

This equation must hold for all values of c, whence

$$\bar{U} = \frac{\lambda^k}{(\xi')^w} U, \quad \bar{V} = \frac{\lambda^k}{(\xi')^w} V.$$

Therefore: *if an absolute covariant be a rational function of its arguments, whose numerator and denominator have no common divisor, the latter are relative covariants of the same degree and weight.*

Let I be an absolute covariant. Then

$$\frac{d\bar{I}}{d\bar{x}} = \frac{1}{\xi'}\frac{dI}{dx},$$

i. e. by differentiating an absolute covariant, a relative covariant may always be obtained of the next higher order. In particular, let Θ_μ, Θ_ν be two *invariants,* of weight μ and ν respectively. Then

$$\mu\,\Theta_\mu\Theta_\nu' - \nu\,\Theta_\nu\Theta_\mu' \tag{24}$$

is a new invariant of weight $\mu + \nu + 1$ which we may, with *Forsyth*, conveniently denote as the *Jacobian* of Θ_μ and Θ_ν.

We shall have occasion to consider a special case of the transformation T, for which

$$\lambda(x) = C(\xi')^\nu,$$

where C is an arbitrary constant, while ν has a fixed constant value; in our case, for example $\nu = \frac{n-1}{2}$. Such transformations form a sub-

group of the infinite group of the transformations T. Let U be a covariant of degree k and of weight w. Then

$$\bar{U} = C^{\lambda}(\xi')^{\lambda\nu - w} U, \quad \bar{y} = C(\xi')^{\nu} y,$$

whence by logarithmic differentiation,

$$\frac{d \log \bar{U}}{d\bar{x}} = \frac{1}{\xi'}\left[\frac{d \log U}{dx} + (k\nu - w)\frac{\xi''}{\xi'}\right],$$

$$\frac{d \log \bar{y}}{d\bar{x}} = \frac{1}{\xi'}\left[\frac{d \log y}{dx} + \nu \frac{\xi''}{\xi'}\right],$$

so that

$$\nu U' y - (k\nu - w) U y' \tag{25}$$

is seen to be a covariant for the sub-group

$$\lambda(x) = C(\xi')^{\nu},$$

if U is a covariant of degree k and of weight w.

§ 4. Canonical form of the differential equation and of its invariants.

From (21), making use of our expressions for $A_{m,m}$, $A_{m,m-1}$, etc., we find

$$\begin{aligned} \bar{p}_1 &= \frac{1}{\xi'}\left[p_1 + \frac{n-1}{2}\eta\right], \quad \text{where} \quad \eta = \frac{\xi''}{\xi'}, \\ \bar{p}_2 &= \frac{1}{(\xi')^2}\left[p_2 + (n-2)p_1\eta + \frac{1}{12}(3n^2 - 11n + 10)\eta^2 + \frac{n-2}{3}\eta'\right], \end{aligned} \tag{26}$$

whence

$$\frac{d\bar{p}_1}{d\bar{x}} = \frac{1}{(\xi')^2}\left[p_1' - p_1\eta - \frac{n-1}{2}\eta^2 + \frac{n-1}{2}\eta'\right]. \tag{26a}$$

According to (9) we shall therefore find

$$\bar{P}_2 = \frac{1}{(\xi')^2}\left[P_2 + \frac{n+1}{12}\eta^2 - \frac{n+1}{6}\eta'\right]. \tag{27}$$

Being a seminvariant, P_2 is not changed by any transformation

$$y = \lambda \bar{y}$$

affecting only the dependent variable. According to (6) and (26), if we make successively the two transformations

$$y = \lambda \bar{y}, \quad \bar{x} = \xi(x),$$

p_1 is changed into

$$\bar{p}_1 = \frac{1}{\xi'}\left[p_1 + \frac{\lambda'}{\lambda} + \frac{n-1}{2}\eta\right]. \tag{28}$$

Suppose that (1) has been reduced to its semi-canonical form, so that $p_1 = 0$. Then, as (28) shows, $\bar{p}_1$ will be zero, if and only if

$$\frac{\lambda'}{\lambda} + \frac{n-1}{2}\frac{\xi''}{\xi'} = 0, \tag{29}$$

i. e. if

$$\lambda = C\left(\frac{d\xi}{dx}\right)^{-\frac{n-1}{2}}. \tag{29a}$$

In other words, *the most general transformation of the form T, which leaves the semi-canonical form invariant, is*

$$\bar{x} = \xi(x), \quad \bar{y} = C(\xi')^{\frac{n-1}{2}} y, \tag{30}$$

where $\xi(x)$ *is an arbitrary function, and* C *an arbitrary constant.*

It is clear from the general theory that the transformations (30) must form a *group*, a sub-group of (2). The group-property may moreover be verified directly.

We may now choose $\xi(x)$ in such a way as to make P_2 vanish. According to (27) it is sufficient for this purpose to take for $\xi(x)$ such a function of x that η shall satisfy the equation

$$\eta' - \frac{1}{2}\eta^2 = \frac{6}{n+1} P_2, \tag{31}$$

which is of the *Riccati* form.

We thus obtain an equation equivalent to (1) for which

$$\bar{p}_1 = \bar{p}_2 = 0.$$

That this transformation is possible was first shown by *Laguerre.* The canonical form of (1) which is thus obtained was employed by *Forsyth*, for the theory of invariants. We shall therefore speak of this reduction, as the reduction to the *Forsyth-Laguerre* canonical form, this form being characterized by the absence of the $n-1^{\text{th}}$ and $n-2^{\text{th}}$ derivatives.

Let us suppose this reduction made, so that $p_1 = p_2 = 0$, and therefore $P_2 = 0$. The most general transformation which leaves the canonical form invariant, must, according to (31), satisfy the further condition

$$\mu = \eta' - \frac{1}{2}\eta^2 = 0.$$

But, if we introduce $\eta = \frac{\xi''}{\xi'}$ into this equation, we find

$$\mu = \frac{\xi^{(3)}}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 = 0.$$

The expression on the left is nothing more or less than the *Schwarzian derivative* of ξ with respect to x. The most general solution of this equation is

$$\xi = \frac{\alpha x + \beta}{\gamma x + \delta},$$

where α, β, γ, δ are constants, whose ratios only are of importance. The relation between λ and ξ is, of course, the same as before.

Therefore, the most general transformation, which leaves the Laguerre-Forsyth canonical form invariant, is

$$(32) \qquad \xi = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \eta = \frac{Cy}{(\gamma x + \delta)^{n-1}}.$$

The totality of these transformations constitutes a four-parameter group.

Let us suppose that (1) has been reduced to the canonical form. Its invariants will assume an exceptionally simple (canonical) form, owing to the vanishing of the coefficients p_1 and p_2. But we have just seen that this reduction may be accomplished in ∞^4 different ways. For any one of these reductions, of course, the absolute invariants of (1) have the same value. But they also have the same form; for, no matter how the reduction has been accomplished, in the resulting canonical form, p_1 and p_2 are zero. The invariants of (1) in their canonical form, must therefore be such functions of the coefficients of the canonical form of (1), as remain invariant under all transformations which leave the canonical form unchanged, i. e. under the transformations (32). On the other hand, any function of the coefficients of the canonical form, which remains invariant under transformations (32), must be the canonical form of an invariant of (1). For, although (1) can be reduced to any one of ∞^4 different canonical forms, this totality of canonical forms is the same for any equation equivalent to (1). A function of the coefficients of the canonical form, which remains unaltered by the transformations (32), has therefore the same significance for (1) as for any equation equivalent to (1), i. e. it is the canonical form of an invariant.

To find the canonical form of the invariants of (1) *is, therefore, the same as to find the invariants of an equation in its canonical form under the transformations* (32).

Let, therefore

$$(33) \qquad y^{(n)} + \binom{n}{3} q_3 y^{(n-3)} + \cdots + q_n y = 0$$

be a linear differential equation in its canonical form. We proceed to determine its invariants under the transformations of the four-parameter group

$$(32) \qquad \bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \bar{y} = \frac{Cy}{(\gamma x + \delta)^{n-1}}.$$

These will be the invariants of the general equation in their canonical form.

We may assume

$$\alpha\delta - \beta\gamma = 1, \tag{32a}$$

since only the ratios of α, β, γ, δ have any significance. It is then evident that (32) contains only four parameters.

To determine the infinitesimal transformations of (32), put

$$\alpha = 1 + c_1\delta t, \quad \beta = c_2\delta t, \quad \gamma = c_3\delta t, \quad \delta = 1 + c_4\delta t, \quad C = 1 + \varepsilon\delta t$$

where δt is an infinitesimal. Then, neglecting higher powers of δt,

$$\delta x = \bar{x} - x = |c_2 + (c_1 - c_4)x - c_3x^2|\,\delta t,$$
$$\delta y = \bar{y} - y = y[\varepsilon - (n-1)(c_3x + c_4)]\,\delta t.$$

But from (32a) we find

$$c_1 + c_4 = 0.$$

We may therefore put

$$c_2 = \alpha_0, \quad c_1 - c_4 = 2\alpha_1, \quad -c_3 = \alpha_2, \quad -c_4 = \alpha_1,$$

so that we obtain the following, as the infinitesimal transformations of x and y,

$$\delta x = (\alpha_0 + 2\alpha_1 x + \alpha_2 x^2)\,\delta t, \quad \delta y = [\varepsilon + (n-1)(\alpha_1 + \alpha_2 x)]y\,\delta t,$$

or

$$\delta x = \xi\,\delta t, \quad \delta y = \left(\varepsilon + \frac{n-1}{2}\xi'\right)y\,\delta t, \tag{34}$$

if we put

$$\xi = \alpha_0 + 2\alpha_1 x + \alpha_2 x^2.$$

Let f be any function of x, and $\bar{f}$ the corresponding function of $\bar{x}$. Then

$$\frac{d\bar{f}}{d\bar{x}} = \frac{d\bar{f}}{dx}\frac{dx}{d\bar{x}}.$$

Since we have

$$\bar{x} = x + \xi\,\delta t, \quad \frac{d\bar{x}}{dx} = 1 + \xi'\,\delta t, \quad \bar{f} = f + \delta f,$$

we find

$$\frac{d\bar{f}}{d\bar{x}} = \left(\frac{df}{dx} + \frac{d(\delta f)}{dx}\right)\left(1 - \xi'\,\delta t\right) = \frac{df}{dx} + \frac{d(\delta f)}{dx} - \xi'\frac{df}{dx}\,\delta t,$$

i. e.

$$\delta(f') = \frac{d}{dx}(\delta f) - \xi' f'\,\delta t. \tag{35}$$

If we apply this formula, we shall find

$$\delta(y^{(k)}) = \left[\left(\varepsilon + \frac{n-2k-1}{2}\xi'\right)y^{(k)} + \frac{k(n-k)}{2}y^{(k-1)}\xi''\right]\delta t, \tag{36}$$
$$(k = 0, 1, 2, \ldots n).$$

To find the result of the infinitesimal transformation upon (33) we must substitute in it,

$$y^{(k)} = \frac{d^k \bar{y}}{d\bar{x}^k} - \delta(y^{(k)}).$$

If we denote $\frac{d^k \bar{y}}{d\bar{x}^k}$ by $y^{(k)}$, (33) becomes, after this substitution,

$$\left[1-\left(\varepsilon-\frac{n+1}{2}\xi'\right)\delta t\right]y^{(n)}+\sum_{k=1}^{n}\left\{-\binom{n}{n-k}q_{n-k}\frac{k(n-k)}{2}\xi''\delta t\right.$$
$$\left.+\binom{n}{n-k+1}q_{n-k+1}\left[1-\left(\varepsilon+\frac{n-2k+1}{2}\xi'\right)\delta t\right]\right\}y^{(k-1)}=0,$$

or

$$\overline{y^{(n)}}+\sum_{k=1}^{n}\left\{\binom{n}{n-k+1}q_{n-k+1}+\left[-\binom{n}{n-k+1}q_{n-k+1}\left(\varepsilon+\frac{n-2k+1}{2}\xi'\right)\right.\right.$$
$$-\binom{n}{n-k}\frac{k(n-k)}{2}q_{n-k}\xi''$$
$$\left.\left.+\binom{n}{n-k+1}q_{n-k+1}\left(\varepsilon-\frac{n+1}{2}\xi'\right)\right]\delta t\right\}\overline{y^{(k-1)}}=0.$$

If, therefore, we denote the coefficient of $\bar{y}^{(k)}$ by $\binom{n}{n-k}q_{n-k}$, and put

$$\overline{q_{n-k}}-q_{n-k}=\delta q_{n-k},$$

we shall find

$$\delta q_{n-k}=-\left[(n-k)\xi' q_{n-k}+\frac{(n-k)(n-k-1)}{2}\xi'' q_{n-k-1}\right]\delta t$$
$$(k=1, 2, \ldots n-1),$$

or

$$\frac{\delta q_i}{\delta t}=-i\xi' q_i-\frac{i(i-1)}{2}\xi'' q_{i-1}.$$

The continued application of (35) will then give, by induction,

(37) $$\frac{\delta q_i^{(k)}}{\delta t}=-(i+k)\xi' q_i^{(k)}-\frac{1}{2}\left[k(k+2i-1)q_i^{(k-1)}+i(i-1)q_{i-1}^{(k)}\right]\xi'',$$
$$(i=3, 4, \ldots n;\ k=0, 1, 2, \ldots).$$

Let f be a function of $y, y', \ldots, y^{(l)} \ldots$ and of the quantities $q_i^{(j)}$ up to weight w, so that $i+j \leq w$, and $l \leq w$, if $w<n$. If $w>n$ we shall have $i+j<w$, $l<n-1$, since we shall always assume that the higher derivatives have been expressed in terms of $y, y', \ldots y^{(n-1)}$ by means of the differential equation. If then we take $w>n$ we shall have to consider the variables $y, y', \ldots y^{(n-1)}$, and $q_i^{(k)}$ where $i+k<w$, together $(n-2)w-\frac{n^2-3n-2}{2}$ variables. If f be an absolute invariant, containing these variables, we must have $\delta f=0$, i. e.

$$\sum \frac{\partial f}{\partial y^{(l)}} \delta y^{(l)} + \sum \frac{\partial f}{\partial q_i^{(k)}} \delta q_i^{(k)} = 0,$$

for all values of ε, ξ', ξ''. We thus find the following system of partial differential equations for the absolute invariants and covariants

$$\begin{aligned}
&0 = \mathfrak{X}_0^{(w)} f = \sum_{k=0}^{n-1} y^{(k)} \frac{\partial f}{\partial y^{(k)}},\\
(38)\qquad &0 = \mathfrak{X}_1^{(w)} f = \sum_{k=1}^{n-1} k y^{(k)} \frac{\partial f}{\partial y^{(k)}} + \sum_{j=3}^{n} \sum_{k=0}^{w-j} (k+j) q_j^{(k)} \frac{\partial f}{\partial q_j^{(k)}},\\
&0 = \mathfrak{X}_2^{(w)} f = \sum_{k=1}^{n-1} k(n-k) y^{(k-1)} \frac{\partial f}{\partial y^{(k)}} - A_2^{(w)} f,
\end{aligned}$$

where

$$\begin{aligned}
(39)\qquad A_2^{(w)} f = &\sum_{j=3}^{n} \sum_{k=1}^{w-j} k(k+2j-1) q_j^{(k-1)} \frac{\partial f}{\partial q_j^{(k)}}\\
&+ \sum_{j=4}^{n} \sum_{k=0}^{w-j} j(j-1) q_{j-1}^{(k)} \frac{\partial f}{\partial q_j^{(k)}}.
\end{aligned}$$

The three equations (38) are independent, and according to the general theory, form a complete system. Therefore there are

$$(n-2)w - \frac{n^2-3n-2}{2} - 3 = (n-2)w - \frac{n^2-3n+4}{2}$$

absolute invariants and covariants involving quantities of weight no higher than w, where $w \geqq n$. Of these, $n-1$ are necessarily covariants, while all others may be taken as invariants. For, if we assume that f is independent of $y, y', \ldots y^{(n-1)}$, (38) reduces to a system of two equations with n variables less than before. This system must therefore have $n-1$ solutions less than (38), whence our conclusion that all of the solutions of (38) except $n-1$ may be taken as invariants. Of the $n-1$ covariants, all but two may be chosen as being independent of the quantities $q_i^{(k)}$. In fact, the complete system obtained by assuming that f is independent of $q_i^{(k)}$, contains n variables and three equations, so that there are $n-3$ such solutions. Therefore, $n-3$ of the covariants, the so-called *identical covariants* according to *Forsyth*, are the same for all equations of the n^{th} order, while two of them depend upon the coefficients of the equation. For the latter two we may take

$$(40)\qquad \frac{6q_3 y' + \frac{(n-1)q_3' y}{4}}{y q_3^{3}} \quad \text{and} \quad \frac{(n-1)yy'' - \frac{(n-2)(y')^2}{2}}{y^2 q_3^{3}};$$

for we can easily verify that these are solutions of (38). The first equation of (38) merely requires that f shall be homogeneous of

degree zero in y, y', etc.... The second equation requires f to be isobaric of weight zero. q_3 and y are obviously solutions of the last equation of (38). If, therefore, we take any function, homogeneous of degree m and isobaric of weight w, which satisfies the last of the equations (38), we can find from it a solution of (38) by dividing by

$$y^m \cdot q_3^{\frac{w}{3}}.$$

For, such a quotient will obviously satisfy the first two conditions. It will also satisfy the last since the quotient of two solutions of $\mathfrak{X}_2^{(n)} f = 0$ will be again a solution. But the numerators of the two expressions (40) are such homogeneous and isobaric functions which verify the equation $\mathfrak{X}_2^{(n)} f = 0$, so that our assertion is proved.

It remains to find the $n-3$ identical covariants and the invariants. We can establish first, the existence of a system of quadratic covariants. Put

$$U_{2j} = \sum_{i=0}^{j-1} \beta_i y^{(2j-i)} y^{(i)} + \frac{1}{2} \beta_j [y^{(j)}]^2, \tag{41}$$

where β_i are constants. This expression is homogeneous of degree two, and isobaric of weight $2j$. We shall be able to determine the coefficients β_k so as to have

$$\mathfrak{X}_2^{(n)} U_{2j} = 0.$$

In fact

$$\mathfrak{X}_2^{(n)} U_{2j} = \sum_{k=1}^{j} [k(n-k)\beta_k + (2j-k+1)(n-2j+k-1)\beta_{k-1}] y^{(k-1)} y^{(2j-k)}$$

This will be zero, if we put

$$\beta_k = -\frac{(2j-k+1)(n-2j+k-1)}{k(n-k)} \cdot \beta_{k-1}, \quad \beta_0 = 1,$$

whence

$$\beta_k = (-1)^k \frac{(2j)!\,(n-2j+k-1)!\,(n-k-1)!}{(2j-k)!\,k!\,(n-2j-1)!\,(n-1)!}, \quad (k = 0, 1, 2, \ldots j) \tag{42}$$

For $j = 1$, we find

$$U_{21} = \frac{U_2}{n-1} = y''y - \frac{n-2}{n-1}(y')^2$$

if

$$U_2 = (n-1) y'' y - (n-2)(y')^2. \tag{43}$$

If we put

$$\Phi_{2j} = \frac{y^{2j-2} U_{2j}}{U_2^j}, \quad j = \begin{cases} 2, 3, \ldots \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ 2, 3, \ldots \frac{n-1}{2} & \text{,, ,, ,, odd} \end{cases}, \tag{44}$$

we have $\frac{n}{2} - 2$ identical covariants or $\frac{n-3}{2}$ according as n is even or odd.

From these covariants we can deduce the others. We have

$$\frac{\delta U_{2j}}{\delta t} = 2\left(\varepsilon + \frac{n-2j-1}{2}\xi'\right)U_{2j},$$

$$\frac{\delta y}{\delta t} = \left(\varepsilon + \frac{n-1}{2}\xi'\right)y,$$

whence

$$\frac{\delta U'_{2j}}{\delta t} = 2\left(\varepsilon + \frac{n-2j-2}{2}\xi'\right)U'_{2j} + (n-2j-1)\xi'' U_{2j},$$

$$\frac{\delta y'}{\delta t} = \left(\varepsilon + \frac{n-3}{2}\xi'\right)y' + \frac{n-1}{2}y\xi''.$$

Put

$$U_{2j+1} = (n-1)yU'_{2j} - 2(n-2j-1)y'U_{2j}.$$

We find

$$\frac{\delta U_{2j+1}}{\delta t} = \left[3\varepsilon + \frac{3(n-1)}{2} - (2j+1)\right]U_{2j+1},$$

so that U_{2j+1} is a relative covariant of degree 3 and weight $2j+1$. This same result might have been obtained by applying the general formula (25). Therefore

$$(45) \qquad \Phi_{2j+1} = \frac{y^{2j-2}U_{2j+1}}{U_2^{\frac{2j+1}{2}}}, \quad \left(j = 1, 2, 3, \ldots \frac{n}{2} - 1 \text{ or } \frac{n-3}{2}\right)$$

gives $\frac{n}{2} - 1$ or $\frac{n-3}{2}$ further identical covariants according as n is even or odd. We have found explicit expressions for the $n-3$ identical covariants. For it is evident that Φ_3, Φ_4, etc. are independent, since, taken in this order, Φ_k is the first which involves $y^{(k)}$.

We now proceed to compute the invariants. The first equation of (38) becomes superfluous. The second is satisfied by any function of the quantities $q_i^{(k)}$ isobaric of weight zero. We shall, therefore, seek isobaric solutions of

$$A_2^{(n)} f = 0,$$

and then, by division with an appropriate power of q_3, deduce therefrom an absolute invariant.

There are $n-2$ relative invariants which are linear in the quantities $q_i^{(k)}$. In fact, let us put

$$(46) \qquad \Theta_m = \sum_{s=0}^{m-3} \alpha_{m,s} q_{m-s}^{(s)}, \quad (m = 3, 4, \ldots n).$$

We shall find

$$A_2^{(n)}\Theta_m = \sum_{j=1}^{m-1} [(m-j)(m+j-1)\alpha_{m,m-j} + j(j+1)\alpha_{m,m-j-1}]q_j^{(m-j-1)},$$

so that Θ_m is a solution of $A_2^{(n)} f = 0$, if

$$\alpha_{m,m-j} = -\frac{j(j+1)}{(m-j)(m+j-1)}\alpha_{m,m-j-1}, \quad (j = 3, 4, \ldots m-1)$$

or

$$\alpha_{m,s} = -\frac{(m-s)(m-s+1)}{s(2m-s-1)}\alpha_{m,s-1}, \quad (s = 1, 2, \ldots m-3),$$

whence

$$\text{(47)} \qquad \alpha_{m,s} = (-1)^s \frac{(m-2)!\,m!\,(2m-s-2)!}{(m-s-1)!\,(m-s)!\,(2m-3)!\,s!}\,\frac{\alpha_{m,0}}{2},$$

which equation is satisfied also for $s = 0$. Put $\alpha_{m,0} = 2$, so that

$$\text{(48)} \qquad \Theta_m = \sum_{s=0}^{m-3}(-1)^s \frac{(m-2)!\,m!\,(2m-s-2)!}{(m-s-1)!\,(m-s)!\,(2m-3)!\,s!}\,q_{m-s}^{(s)},$$

$$(m = 3, 4, \ldots n).$$

This gives us $n-2$ relative invariants, of which the first is simply

$$\Theta_3 = q_3,$$

so that the functions

$$\text{(49)} \qquad J_m = \frac{\Theta_m}{q_3^{\frac{m}{3}}}, \quad (m = 4, 5, \ldots n)$$

represent $n-3$ absolute invariants.

We may easily verify that

$$\text{(50)} \qquad J_3 = \frac{6q_3''q_3 - 7(q_3')^2}{q_3^{\frac{8}{3}}}$$

is a further absolute invariant. These $n-2$ absolute invariants are independent, and the remaining invariants,

$$(n-2)(w-1) - \frac{1}{2}(n^2-n+2)$$

in number can be derived from these by differentiation. In fact, if J is an absolute invariant, so is

$$q_3^{-\frac{1}{3}}\frac{dJ}{dx}.$$

If therefore we denote the operator

$$q_3^{-\frac{1}{3}}\frac{d}{dx}$$

by ϑ, we shall have the following additional absolute invariants

$$\text{(51)} \qquad \begin{aligned} &\vartheta J_m,\ \vartheta^2 J_m, \ldots \vartheta^{w-m}J_m; \quad (m = 4, 5, \ldots n),\\ &\vartheta J_3,\ \vartheta^2 J_3, \ldots \vartheta^{w-5}J_3, \end{aligned}$$

which are independent and

$$w-5+[w-4+w-5+w-6+\cdots+w-n]$$
$$=(n-2)(w-1)-\frac{1}{2}(n^2-n+2)$$

in number, so that we have them all. That they are indeed independent may be seen as follows. If there were a relation between them, it would be a relation between the quantities $J_3, \ldots J_m$ and their derivatives up to a certain order, no higher than $w-4$. Solve this relation for one of the derivatives of the highest order which occurs in it, so that we shall have *identically* (i. e. for all values of p_k, p_k', p_k'', etc.)

$$\frac{d^\lambda J_m}{dx'} = f(J_k, J_k', \ldots J_k^{(\lambda)}).$$

Since the left member is a total derivative, so is the right member, and integration would give rise to a relation between the derivatives of order $\lambda - 1$. Continuing this process would give finally a relation between $J_3, J_4, \ldots J_n$. But these are independent. We have therefore found the functionally complete system of invariants and covariants in their canonical form.

The numerator of ϑJ_m may be written

$$3\Theta_3\Theta_m' - m\Theta_m\Theta_3'$$

a combination which we have decided to call, with *Forsyth*, the *Jacobian* of Θ_3 and Θ_m.

Our result in regard to invariants, may therefore be expressed as follows. *All relative invariants may be derived from the linear invariants Θ, .. Θ_n and from $\Theta_{3\cdot 1} = 6q_3''q_3 - 7(q_3')^2$, by first combining Θ_3 with all of the others by the Jacobian process, then combining Θ_3 in the same way with the resulting new invariants, etc...*

An invariant of (1) in its general form can contain the coefficients $p_1, p_2, \ldots p_n$ of (1) only in the seminvariant combinations $P_2, P_3 \ldots P_n, P_2' \ldots P_n'$, etc, and must be an isobaric function of these quantities. If we form such an invariant of weight m in its general form, it will contain certain terms of the first degree, certain terms of the second degree, and so on But by a transformation of the form T, we can reduce the equation to the canonical form, which is characterized by the conditions $P_2=0$, $P_3=q_3, \ldots P_n=q_n$ If

$$f(P_2, P_3 \ldots P_n;\ P_2' \ldots P_n';\ \ldots)$$

is the general form of the invariant, its canonical form becomes

$$f(0, q_3 \ldots q_n;\ 0, q_3' \ldots q_n';\ \ldots),$$

so that all of the terms of such an invariant in its uncanonical form, except those which contain $P_2, P_2', P_2'', \ldots$ as factor, may be obtained

by substituting P_k in place of q_k. If we continue to denote by Θ_m the invariant, which in its canonical form reduces to the expression which we have computed, we see that the linear terms of Θ_m, excepting a possible term of the form $P_2^{(m-2)}$, can be obtained by putting $q_k = P_k$ $(k = 3, 4, \dots n)$ in the formulae which give explicitly its canonical form. We shall continue also, with *Forsyth*, to speak of these invariants as linear invariants.

The linear invariants in their uncanonical form contain, beside those terms which have been determined explicitly, others which have $P_2, P_2', \dots$ as factors. Are these terms also expressible as integral rational functions of $P_2, P_3, \dots P_n$ and of their derivatives?

We observe in the first place that, if the formulae expressing $P_k^{(i)}$ in terms of $P_k^{(i)}$ be derived from (21), these are linear in $P_k^{(i)}$, and the coefficients A_k, are algebraic functions of the derivatives of ξ. The invariant equations could clearly be obtained by eliminating these derivatives of ξ from the equations. It must therefore be possible, by algebraic elimination, to set up a complete system of invariants, each of which is algebraic in the variables involved. We shall speak of these as the algebraic invariants, so as to distinguish them from those whose canonical form we have calculated, and which may be called the fundamental invariants. Since both systems of invariants are complete, it must be possible to express the algebraic invariants as functions of the fundamental invariants and vice-versa. For the canonical form, we know that the fundamental invariants are themselves algebraic, and therefore expressible as *algebraic* functions of the algebraic invariants. But a relation between invariants is not changed by any transformation of the form T, such as the reduction to the canonical form. Therefore, the fundamental invariants are always algebraic functions of the algebraic invariants, i. e. they are themselves algebraic.

We may, therefore, assume that Θ_i is a root of an irreducible algebraic equation

$$(52) \qquad a_r \Theta_i^r + a_{r-1} \Theta_i^{r-1} + \dots + a_1 \Theta_i + a_0 = 0,$$

where $a_0, a_1, \dots a_r$ are integral rational functions of $P_2, P_3, \dots P_n$ and of the derivatives of these quantities. After an arbitrary transformation of the form T, (1) is converted into a differential equation, whose coefficients may be denoted by $\bar{p}_k$. If we denote by $\bar{a}_k$ and $\bar{\Theta}_i$ the same function of these quantities $\bar{p}_k$ as a_k and Θ_i are of the quantities p_k, $\bar{\Theta}_i$ must satisfy the equation

$$\bar{a}_r \bar{\Theta}_i^r + \bar{a}_{r-1} \bar{\Theta}_i^{r-1} + \dots + \bar{a}_1 \bar{\Theta}_i + \bar{a}_0 = 0.$$

On the other hand, if Θ_i is of weight ν, we know that

$$\Theta_\nu^k = \frac{1}{(\xi')^{\nu k}} \Theta_\nu^k,$$

where ξ' is an arbitrary function of x, so that

$$\frac{a_r \Theta_\nu^r}{(\xi')^{\nu r}} + \frac{\bar{a}_{r-1} \Theta_\nu^{r-1}}{(\xi')^{\nu(r-1)}} + \cdots + \frac{\bar{a}_1 \Theta_\nu}{(\xi')^\nu} + a_0 = 0. \tag{53}$$

The equations (52) and (53) must be identical. Otherwise Θ_ν would satisfy an equation of the same form but of lower degree. Therefore, the coefficients a_k of (52) must be invariants.

For the canonical form however, Θ_ν becomes an integral rational function of $P_k, P_k', \ldots$ etc. On reduction to the canonical form, the equation (52) must therefore reduce to the form

$$a_1 \Theta_\nu + a_0 = 0,$$

where a_1 is merely a numerical factor, and a_0 an integral rational invariant. But again, since the reduction to the canonical form cannot change a relation between invariants, this same equation must be true in general

Therefore, *the fundamental invariants whose canonical form has been calculated, are in their uncanonical form integral rational invariants.*

We may now conclude that *the non-linear part of the linear invariant Θ_m cannot contain P_m or even P_{m-1}*, since each of its terms must contain P_2 or a derivative of P_2 as a factor, and its weight must be equal to m. This remark will be of importance shortly.

In our complete system of invariants we have employed one, whose canonical form is $6q_3q_3'' - 7(q_3')^2$. It is one of a system, whose general form we shall now deduce.

Consider an invariant Θ_m of weight m Then, after the transformation $\xi = \xi(x)$,

$$\Theta_m = \frac{1}{\xi'^m} \Theta_m,$$

whence

$$\frac{d \log \Theta_m}{d\xi} = \frac{1}{\xi'}\left[\frac{d \log \Theta_m}{dx} - m\eta\right], \quad \eta = \frac{\xi''}{\xi'},$$

and

$$\frac{d^2 \log \Theta_m}{d\xi^2} = \frac{1}{(\xi')^2}\left[\frac{d^2 \log \Theta_m}{dx^2} - \eta \frac{d \log \Theta_m}{dx} + m\eta^2 - m\eta'\right].$$

Between these two equations, eliminate $\frac{d \log \Theta_m}{dx}$. We find

$$X = \frac{1}{(\xi')^2}\left[X + m^2\eta^2 - 2m^2\eta'\right],$$

where

$$X = 2m \frac{d^2 \log \Theta_m}{dx^2} - \left(\frac{d \log \Theta_m}{dx}\right)^2.$$

But we have also

$$P_2 = \frac{1}{(\xi')^2}\left[P_2 + \frac{n+1}{12}\eta^2 - \frac{n+1}{6}\eta'\right],$$

so that

$$X - \frac{12m^2}{n+1}P_2$$

is an invariant. The numerator of this expression, when reduced to a fractional form, we denote by $\Theta_{m\cdot 1}$. It is equal to

$$(54) \qquad \Theta_{m\cdot 1} = 2m\Theta_m\Theta_m'' - (2m+1)(\Theta_m')^2 - \frac{12m^2}{n+1}P_2\Theta_m^2,$$

and is called by *Forsyth*, the *quadriderivative* of Θ_m. Its weight is $2(m+1)$ For $m = 3$ we get an invariant which, in its canonical form, coincides with

$$6q_3q_3'' - 7(q_3')^2.$$

It is now clear that, *if $\Theta_3, \Theta_4, \ldots \Theta_n$ and $\Theta_{3\cdot 1}$ are given as functions of x, the coefficients of the semi-canonical form $P_2, P_3, \ldots P_n$ can be expressed in terms of them and of their derivatives. provided that* $\Theta_3 \neq 0.$ [1])

Upon this theorem a new proof may be founded of the fact that all invariants can be obtained from these fundamental ones by the Jacobian process. We shall not insist upon this. We shall show, however, that our system of fundamental invariants, together with the Jacobian process, furnishes a complete system of invariants in a more special sense Not only can any rational invariant be expressed as *a function* of these invariants, (this we have already shown), but as *a rational function*

Since the quantities $P_2, P_3, \ldots P_n$ can be expressed rationally in terms of $\Theta_3, \ldots \Theta_n$ and $\Theta_{3\cdot 1}$, and of the derivatives of these quantities, any invariant which is a rational function of the semi-invariants P_k, P_k', etc. becomes a rational function of these $n-1$ fundamental invariants and of their derivatives The numerator and denominator of this rational function must themselves be invariants. We shall show that, except for a factor of the form Θ_3^λ, every invariant, integral, rational function of this form may be converted into an integral rational function of the fundamental invariants, i. e. of $\Theta_3 \ldots \Theta_n$, $\Theta_{3\cdot 1}$ and of the Jacobians of Θ_3 with the others

In order to prove this, it is clearly permissible to make use of the canonical form, since a relation between invariants is not changed by such a reduction. Let

1) This follows from the expressions for the linear part of the linear invariants, together with the remark that the non-linear part of Θ_m does not contain P_m.

$$\varphi = \psi + \chi$$

be an invariant, integral and rational, of weight w, containing no higher derivatives of $\Theta_3 \ldots \Theta_m$, $\Theta_{3\,1}$ than the m^{th}, and let ψ represent the aggregate of those terms whose degree μ in the m^{th} derivatives is the highest. We shall then have

$$(55) \qquad \psi = \Sigma A_{r_3 \ldots r_n r_8'} \Theta_3^{(m) r_3} \ldots \Theta_n^{(m) r_n} \Theta_{3\,1}^{(m) r_8'},$$
$$r_3 + r_4 + \cdots + r_n + r_8' = \mu,$$

where we have denoted the exponent of $\Theta_{3\,1}^{(m)}$ by r_8' because $\Theta_{3\,1}$ is of weight 8. The other term χ of φ will contain the derivatives of the m^{th} order only to a degree lower than μ, and the coefficients A of ψ can depend only upon derivatives of order lower than m.

The transformation $\bar{x} = \xi(x)$, $\bar{y} = \lambda(x) y$ converts $\Theta_i^{(m)}$ into $\bar{\Theta}_i^{(m)}$, where

$$\bar{\Theta}_i = \frac{1}{(\xi')^i} \Theta_i, \quad \bar{\Theta}_i^{(m)} = \frac{1}{(\xi')^{i+m}} \left[\Theta_i^{(m)} + \text{terms of lower order} \right],$$

while

$$(56) \qquad \bar{\varphi} = \frac{1}{(\xi')^w} \varphi.$$

Let $\bar{A}$ be the new value of A. Then, the general term of $\bar{\psi}$ will become

$$\frac{\bar{A}_{r_3 \ldots r_n r_8'} \Theta_3^{(m) r_3} \ldots \Theta_n^{(m) r_n} \Theta_{3\,1}^{(m) r_8'}}{(\xi')^{(3+m) r_3 + \cdots + (n+m) r_n + (8+m) r_8'}}$$

plus terms of lower degree in the derivatives of highest order.

But, on the other hand, (56) shows that the general term of $\bar{\psi}$ will be

$$\frac{1}{(\xi')^w} A_{r_3 \ldots r_n r_8'} \Theta_3^{(m) r_3} \ldots \Theta_n^{(m) r_n} \Theta_{3\,1}^{(m) r_8'}.$$

These two expressions must be identical, since the expression of φ in terms of these quantities is obviously unique. Therefore $\bar{A}$ and A are identical except for a power of ξ', i. e. the coefficients A must be invariants.

As has been remarked, we may assume that (1) has been reduced to its canonical form. Since, in that case,

$$\Theta_{3\,1} = 6\Theta_3\Theta_3'' - 7(\Theta_3')^2, \quad \Theta_3'' = \frac{\Theta_{3\,1} + 7\,\Theta_3'^2}{6\,\Theta_3},$$

Θ_3'', $\Theta_3^{(3)}, \ldots \Theta_3^{(m)}$ may be expressed rationally as functions of

$$\Theta_{3\,1},\ \Theta_{3\,1}',\ \ldots\ \Theta_{3\,1}^{(m-2)}$$

and of Θ_3 and Θ_3'. In these expressions the denominators will be mere powers of Θ_3. If we introduce these expressions into φ, it will again assume the form $\varphi = \psi + \chi$, where

$$\psi = \Sigma A\, \Theta_4^{(m)r_4} \dots \Theta_n^{(m)r_n}\, \Theta_{3\cdot 1}^{(m)r_8'}, \quad r_4 + \dots + r_n + r_8' = \mu, \quad m > 1,$$

where the coefficients A will depend upon derivatives of order lower than m but may contain as denominator a power of Θ_3, and where χ is of degree lower than μ in the derivatives of highest order. We conclude as before that the coefficients A are invariants.

Consider the Jacobians

$$(57) \quad \begin{aligned} s_{1\nu} &= 3\Theta_3\Theta_\nu' - \nu\Theta_\nu\Theta_3', & t_1 &= 3\Theta_3\Theta_{3\cdot 1}' - 8\Theta_{3\cdot 1}\Theta_3', \\ s_{2\nu} &= 3\Theta_3 s_{1\nu}' - (\nu+4)\Theta_{1\nu}\Theta_3', & t_2 &= 3\Theta_3 t_1' - 12 t_1\Theta_3', \\ &\dots\dots & &\dots\dots \\ s_{m\nu} &= 3\Theta_3 s_{m-1,\nu}' - (\nu+4m-4)s_{m-1,\nu}\Theta_3', & & \\ & & t_m &= 3\Theta_3 t_{m-1}' - 4(m+1)t_{m-1}\Theta_3', \\ (\nu &= 4, 5, \dots m) & & \end{aligned}$$

all of which are invariants Clearly

$$\varphi - \sum \frac{A s_{m4}^{r_4} \dots s_{mn}^{r_n} t_m^{r_8'}}{(3\Theta_3)^{m(r_4 + \dots + r_n + r_8')}} = \varphi_1$$

will be an invariant, whose degree in the highest derivatives is no higher than $\mu - 1$. By continuing this process upon φ_1, we shall finally obtain for φ an expression of the form

$$\varphi = \sum \frac{A s_{m4}^{r_4} \dots s_{mn}^{r_n} t_m^{r_8'}}{\Theta_3^R} + \sum \frac{B s_{m4}^{\lambda_4} \dots s_{mn}^{\lambda_n} t_m^{\lambda_8'}}{\Theta_3^\Lambda} + \dots + F,$$

where

$$\begin{gathered} r_4 + \dots + r_n + r_8' = \mu, \\ \lambda_4 + \dots + \lambda_n + \lambda_8' = \mu - 1, \\ \dots\dots\dots \end{gathered}$$

and where the coefficients $A, B, \dots F$ are integral rational invariants containing only derivatives of order lower than m. Each of these may be reduced in like fashion until we get an expression for φ in terms of the Jacobians

$$s_{m\nu}, \; s_{m-1,\nu}, \dots s_{2,\nu}; \quad t_m, \; t_{m-1}, \dots t_2;$$

whose coefficients contain only the first derivatives of $\Theta_3 \dots \Theta_n, \Theta_{3\cdot 1}$. In this case we cannot, as in the others, remove the term Θ_3'. But, if

$$(58) \qquad \Sigma C (\Theta_4')^{r_4} \dots (\Theta_n')^{r_n} (\Theta_{3\cdot 1}')^{r_8'}$$

is such an invariant expression, we know that it can be only a function of the Jacobians $s_{1\nu}$ and t_1. If therefore, by means of (57), we express $\Theta_4', \dots \Theta_n', \Theta_{3\cdot 1}'$ in terms of these Jacobians, the terms in Θ_3' must in the aggregate disappear from (58), so that it assumes the form

$$\Sigma D s_{14}^{r_4} \dots s_{1n}^{r_n} t_1^{r_4'},$$

where the coefficients D are functions of $\Theta_3 \dots \Theta_n, \Theta_{3\cdot 1}$ only, and moreover rational functions, since φ is a rational invariant.

We have shown, therefore, that the fundamental system of invariants, which we have determined, is complete in the more restricted sense that every rational invariant is a rational function of the fundamental invariants. For $n = 3$ and for $n = 4$ this theorem was proved by *Halphen*, using the notation of differential invariants, whose relation to the invariants which we are considering will appear later on.

We shall conclude this paragraph with a few remarks of a historical nature. In 1862, *Cockle* started a series of papers[1]) in which he deduced, by finite transformations, a number of the results which we have found. He found the seminvariants, essentially by the method which we have adopted, as well as the semi-covariants, without proving, however, the completeness of the system. He also found one function, invariant under transformations of the independent variable alone. In 1879 *Laguerre*[2]) found the invariant Θ_3 of the equation of the 3[d] order, and showed that its vanishing is the condition for a homogeneous quadratic relation between its solutions, a result which we shall verify later. In a letter to *Laguerre*, *Brioschi*[3]), in the same year, extended *Laguerre*'s investigation to equations of the fourth order. He notices that the form of the invariants is the same for both cases, owing to the fact that he uses what we have called the *Laguerre-Forsyth* canonical form, for which the linear invariants Θ_m are independent of the order n of the equation, as we have seen in general. He also notices that if the invariant Θ_3 vanishes in the case $n = 3$, or if both Θ_3 and Θ_4 vanish in the case $n = 4$, the solutions of the equation are the second and third powers respectively of the solutions of an equation of the second order. He found later, in 1890, that, if all of the linear invariants of an equation of the n^{th} order vanish, the general integral is a binary form of the $n - 1^{\text{th}}$ degree formed from the two solutions of a linear differential equation of the second order. He also found that if only the linear invariants of odd weight vanish, the equation coincides with its *Lagrange* adjoint[4]) These results we shall verify in the next paragraph.

In 1878 *Halphen* published his thesis on the *differential invariants of plane curves*, and in 1880, his paper on the *differential invariants of space curves*[5]) These differential invariants are entirely different

1) *Cockle*, Mostly in the Phil Mag 1862—75

2) *Laguerre*, Comptes Rendus, vol 88 (1879) pp. 116—119 and pp 224—227.

3) *Brioschi*, Société Math. de France Bulletin, vol 7 (1879) pp. 105—108.

4) *Brioschi*, Acta Mathematica, vol 14 (1890) pp. 233—248.

5) *Halphen*, Sur les invariants différentiels. Thèse, Paris 1878. Sur les invariants différentiels des courbes gauches Journal de l'École Polyt. vol 47 (1880).

in form from the invariants of *Laguerre* and *Brioschi*, but can be identified with them, for $n = 3$ and $n = 4$, as was shown by *Halphen* himself. He also proved that his system of invariants is complete in the sense of our last theorem. These papers, geometrical in nature, will occupy us fully later on. In his prize memoire of 1882, however, *Halphen* formally entered the field which we are now discussing. He there considered the invariants of a linear differential equation, and applied them to the problem of determining whether a given equation may be reduced to one of certain types whose integrals are known.[1]) In 1888, *Forsyth*[2]), by the method of infinitesimal transformations, computed all of the invariants in their canonical form, and some of those of lower weight in their general form. *Bouton*[3]), by the application of *Lie*'s general theory to the calculation of these invariants, gave in 1899, a clearer presentation of the subject. I have preserved many of *Bouton*'s notations. The reader will find there also, a detailed appreciation of *Cockle*'s work, as well as further historical remarks. Finally, in 1900, *Fano* published a paper[4]), of which the theory of invariants constitutes only a part, but which explains well the relation of this theory to other branches of the theory of linear differential equations, and which also gives an excellent account of the history of the subject.

§ 5 The Lagrange adjoint equation.

Write (1) in the form

$$(59) \qquad f(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0,$$

so that

$$(60) \qquad a_0 = 1 \quad a_k = \binom{n}{k} p_k .$$

We shall show that there exist certain functions z of x, such that the product

$$z f(y)$$

will be the complete derivative of an expression linear in $y, y', \ldots y^{(n-1)}$. Such a function z may be called an *integrating factor* of (59)

We notice first that integration by parts gives

$$\int z^{(h)} y^{(k-h)} dx = z^{(h)} y^{(k-h-1)} - \int z^{(h+1)} y^{(k-h-1)} dx$$
$$(h = 0, 1, \ldots k-1).$$

1) *Halphen*, Mémoires des Savants Etrangers, vol. 28, 2d series (1882)
2) *Forsyth*, Phil. Trans vol 179 (1888) pp 377 -489.
3) *Bouton*, Am. Journ. of Math. vol 21 (1899) pp. 25—84.
4) *Fano*, Math. Annalen vol 53 (1900) pp 493—590

If we multiply both members of this equation by $(-1)^h$ and form the sum for all values of h from 0 to $k-1$, we shall find

$$\sum_{h=0}^{k-1}(-1)^h\int z^{(h)}y^{(k-h)}\,dx=\sum_{h=0}^{k-1}(-1)^h z^{(h)}y^{(k-h-1)}$$
$$+\sum_{h=0}^{k-1}(-1)^{h+1}\int z^{(h+1)}y^{(k-h-1)}\,dx,$$

whence

$$\int z y^{(k)}\,dx=\sum_{h=0}^{k-1}(-1)^h z^{(h)}y^{(k-h-1)}+(-1)^k\int z^{(k)}y\,dx.$$

We have from (59)

$$\int z f(y)\,dx=\sum_{k=0}^{n}\int y^{(k)}a_{n-k}z\,dx,$$

whence

(61) $$\int z f(y)\,dx=\sum_{k=0}^{n}\sum_{h=0}^{k-1}(-1)^h y^{(k-h-1)}\frac{d^h\,a_{n-k}z}{dx^h}+\int y g(z)\,dx,$$

where

(62) $$g(z)=\sum_{k=0}^{n}(-1)^k\frac{d^k\,a_{n-k}z}{dx^k}.$$

Equation (61) shows that $zf(y)$ is a complete derivative, if and only if

(63) $$g(z)=0.$$

For, it is clearly impossible, that the complete derivative of any linear function of y, y', y'', etc should be of the form $yg(z)$, if $g(z)$ is different from zero. *The equation* (63) *is known as the Lagrange adjoint of* (59). It was considered for the first time by *Lagrange* We see, therefore, that *every solution of the Lagrange adjoint of a linear differential equation furnishes an integrating factor for this equation*

If we write

(64) $$\psi(y,z)=\sum_{k=0}^{n}\sum_{h=0}^{k-1}(-1)^h y^{(k-h-1)}\frac{d^h(a_{n-k}z)}{dx^h},$$

we notice that this expression is linear of the $n-1^{\text{th}}$ order in z as well as in y. It may be called the *adjoint bilinear expression.* Equation (61) may now be written

(65) $$\int[zf(y)-yg(z)]\,dx=\psi(y,z).$$

If any solution z of the adjoint equation be known, we find as a first integral (containing one arbitrary constant), of (59), the equation

$$\psi(y,z)=const$$

In every case, (65) shows that $zf(y)-yg(z)$ *is an exact derivative, for arbitrary functions* y *and* z.

This property is characteristic of the Lagrange adjoint expression, if we denote as such the left member of the Lagrange adjoint equation. In other words, if, for all possible functions y and z, the expression

$$zf(y)-y\varphi(z)$$

is an exact derivative, where $\varphi(z)$ is a linear differential expression of the n^{th} order in z, $\varphi(z)$ must necessarily be the Lagrange adjoint of $f(y)$, i. e.

$$\varphi(z)=g(z)$$

Let us suppose, in fact, that

$$\int[zf(y)-y\varphi(z)]\,dx=\psi_0(y,z),$$

where $\psi_0(y,z)$ is a differential expression in y and z. By subtraction from (65) we find

$$\int y[\varphi(z)-g(z)]\,dx=\psi(y,z)-\psi_0(y,z),$$

i. e. the derivative of the right member would be a linear function of the derivatives of z multiplied into y. But this is clearly impossible. We have, therefore,

$$\varphi(z)=g(z),\quad \psi_0=\psi+\mathit{const},$$

as we proposed to show.

Suppose now that $g(z)$ were given. Since its adjoint must be such a function $f(y)$ as satisfies (65), and is therefore uniquely determined, we see at once that the relation between $f(y)$ and $g(z)$ is reciprocal. In other words, *if of two expressions the second is the Lagrange adjoint of the first, so is the first of the second.*

If, therefore,

$$g(z)=b_0z^{(n)}+b_1z^{(n-1)}+\cdots+b_nz,$$

we shall have also

$$f(y)=\sum_{k=0}^{n}(-1)^k\frac{d^k(b_{n-k}y)}{dx^k},$$

corresponding to (59) and (62).

We proceed to show how the solutions of the two equations can be expressed in terms of each other. It is upon the form of these expressions that the importance of the adjoint equation, from a geometrical point of view, will be found to rest.

Let $y_1, y_2, \ldots y_n$ be a system of linearly independent solutions of $f(y)=0$, so that the determinant

$$(66)\qquad \Delta = \begin{vmatrix} y_1, & y_1', & \dots & y_1^{(n-1)} \\ y_2, & y_2', & \dots & y_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ y_n, & y_n', & \dots & y_n^{(n-1)} \end{vmatrix}$$

does not vanish identically. Such a system of solutions is called a *fundamental system*, because all other solutions can be expressed in terms of them, as homogeneous functions of the first degree with constant coefficients.[1])

Consider the expression

$$\Theta_i(y) = \frac{1}{\Delta}\left[\frac{\partial \Delta}{\partial y_i} y + \frac{\partial \Delta}{\partial y_i'} y' + \dots + \frac{\partial \Delta}{\partial y_i^{(n-1)}} y^{(n-1)}\right].$$

It is an expression linear and homogeneous in $y, y', \dots y^{(n-1)}$; it vanishes for $y = y_k$ if $k \neq i$, and becomes equal to unity for $y = y_i$. If, therefore, we substitute into it, for y the most general solution

$$y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$$

of $f(y) = 0$, Θ_i will assume the value

$$\Theta_i = C_i$$

The equation

$$\frac{d\Theta_i}{dx} = 0,$$

will, therefore, be satisfied by the most general solution of $f(y) = 0$. But it is of the same order as $f(y) = 0$, and its left member can, therefore, differ from $f(y)$ only by a factor z_i. We shall have

$$(67)\qquad \frac{d\Theta_i(y)}{dx} = z_i f(y).$$

Comparison of the coefficients of $y^{(n)}$, in the two members of this equation, gives

$$(68)\qquad z_i = \frac{\partial \log \Delta}{\partial y_i^{(n-1)}}.$$

But since $z_i f(y)$ is an exact derivative, z_i must be a solution of the adjoint equation. If we give to i all of its values from 1 to n, we find in this manner n solutions $z_1, \dots z_n$, of the adjoint equation which, as we shall see immediately, form a fundamental system. It is customary to say that the system z_i is the adjoint of system y_i.

From (68) we find

$$(68\text{a})\qquad \sum_{\alpha=1}^{n} y_\alpha^{(k)} z_\alpha = 0, \quad \sum_{\alpha=1}^{n} y_\alpha^{(n-1)} z_\alpha = 1, \quad (k = 0, 1, \dots n-2).$$

1) For a proof of these well-known theorems we may refer to *Forsyth's Treatise on Differential Equations*, §§ 71 et sequ.

Put, with *Schlesinger*[1]),

$$s_{\lambda k} = \sum_{\alpha=1}^{n} y_\alpha^{(\lambda)} z_\alpha^{(k)}.$$

Then, these equations may be written

$$s_{00} = s_{10} = \cdots = s_{n-2,0} = 0, \quad s_{n-1,0} = 1.$$

But

$$\frac{d s_{\lambda-1,0}}{dx} = s_{\lambda,0} + s_{\lambda-1,1},$$

$$\frac{d^2 s_{\lambda-2,0}}{dx^2} = s_{\lambda,0} + 2 s_{\lambda-1,1} + s_{\lambda-2,2},$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d^\lambda s_{00}}{dx^\lambda} = s_{\lambda,0} + \binom{\lambda}{1} s_{\lambda-1,1} + \binom{\lambda}{2} s_{\lambda-2,2} + \cdots + s_{0\lambda}.$$

Therefore,

$$s_{\lambda k} = 0 \quad \text{if} \quad \lambda + k < n - 1.$$

If we put $\lambda = n - 1$ in the above equations, we find

$$s_{n-1,0} = 1, \; s_{n-2,1} = -1, \; s_{n-3,2} = +1, \ldots s_{n-k-1,k} = (-1)^k,$$

so that

$$s_{\lambda k} = 0 \qquad \text{for} \quad \lambda + k < n - 1,$$
$$s_{\lambda k} = (-1)^k \quad \text{for} \quad \lambda + k = n - 1.$$

If, in these relations we put $\lambda = 0$, we find

$$\sum_{\alpha=1}^{n} z_\alpha^{(k)} y_\alpha = 0, \quad \sum_{\alpha=1}^{n} z_\alpha^{(n-1)} y_\alpha = (-1)^{n-1}, \quad (k = 0, 1, \ldots n-2),$$

whence, if we write

(69)
$$\Delta_1 = \begin{vmatrix} z_1, & z_2, & \ldots z_n \\ z_1', & z_2', & \ldots z_n' \\ \cdot & \cdot & \cdot \\ z_1^{(n-1)}, & z_2^{(n-1)}, & \ldots z_n^{(n-1)} \end{vmatrix},$$

we obtain

(70)
$$y_i = (-1)^{n-1} \frac{\partial \log \Delta_1}{\partial z_i^{(n-1)}}.$$

We find moreover

(71)
$$\Delta \Delta_1 = 1,$$

so that Δ_1 is finite and different from zero if this is true of Δ. This proves that $z_1, \ldots z_n$ form a fundamental system as was stated before. *Equations* (68) *and* (70) *serve to express the solutions of the two equations in terms of each other.*

1) *Schlesinger, Handbuch der Theorie der linearen Differentialgleichungen;* vol. 1, p. 63.

If now a transformation of the form

$$\bar{y} = \lambda(x) y, \quad \bar{x} = \xi(x)$$

be made, we see easily, that to it corresponds the transformation

$$\bar{z} = \frac{z}{\lambda(x)}, \quad \bar{x} = \xi(x)$$

for the adjoint equation. It follows at once, that *the invariants of the Lagrange adjoint equation are also invariants of the original equation.*

We proceed to determine the exact relation between the invariants of the two equations

We find, from (62), the following equations expressing the coefficients b_i of the *Lagrange adjoint* in terms of the coefficients a_λ of the original equation, viz.

$$b_i = (-1)^n \sum_{\lambda = n-i}^{n-1} (-1)^\lambda \binom{\lambda}{n-i} a_{n-k}^{(\lambda-n+i)}, \quad (i = 1, 2, \ldots n),$$

whence, in particular

$$b_1 = -a_1, \quad b_2 = a_2 - (n-1) a_1'.$$

These latter equations show that *the reduction to the Laguerre-Forsyth canonical form is effected simultaneously for any equation together with its Lagrange adjoint.*

Let us assume this canonical form, and let us recur to our customary notation, by writing

$$a_\lambda = \binom{n}{\lambda} p_\lambda, \quad b_k = \binom{n}{k} r_k.$$

Then we shall have

$$p_1 = r_1 = p_2 = r_2 = 0,$$

$$(72) \qquad \frac{1}{i!} r_i = (-1)^n \sum_{\lambda = n-i}^{n-3} (-1)^\lambda \frac{p_{n-\lambda}^{(\lambda-n+i)}}{(\lambda - n + i)!\,(n-\lambda)!} = \frac{1}{i!} (-1)^i p_i + \cdots$$

where the terms not written depend upon p's of index lower than i.

We can find the canonical form of the linear invariants of the Lagrange adjoint equation, by merely substituting these values for the coefficients r_i in place of q_i in equation (48). The term of highest index present in Θ_m, is q_m. If we put $q_i = r_i$ in Θ_m, and then express r_i in terms of p_i by means of (72), we shall get a linear invariant Σ_m of weight m. The coefficient of p_m in Σ_m will differ from that in Θ_m by the factor $(-1)^m$. But the linear invariant of weight m is uniquely determined up to a constant factor, i. e. the most general linear invariant of weight m is $C\Theta_m$. We must, therefore, have

$$\Sigma_m = (-1)^m \Theta_m,$$

i. e. *the linear invariants of even weight are identical for a linear differential equation and its Lagrange adjoint. The invariants of odd weight for the two equations, differ only in sign.*

If an equation coincides with its adjoint, the invariants of odd weight must vanish. From equations (48) we see conversely, that if they vanish, the equation coincides with its adjoint. For, the invariants of even weight being arbitrary functions of x, this equation for $(m = 3, 4, \ldots n)$ enables us to compute successively and in a unique fashion $q_3 = 0$, $q_4, q_5, \ldots q_n$. We see therefore that the theorem of *Brioschi*, mentioned in the last paragraph, is true *A linear differential equation coincides with its Lagrange adjoint, if and only if its invariants of odd weight vanish.*

If the invariants of even order also vanish, we thus find that the canonical form of our differential equation becomes

$$\frac{d^n \bar{y}}{d\bar{x}^n} = 0.$$

But, in order to reduce (1) to its canonical form, we made the transformation

$$\bar{x} = \xi(x), \quad y = \lambda(x)\bar{y},$$

where $\xi(x)$ was any solution of the equations

$$\eta' - \frac{1}{2}\eta^2 = \frac{6}{n+1}P_2, \quad \eta = \frac{\xi''}{\xi'},$$

and where

$$\lambda = (\xi')^{-\frac{n-1}{2}}.$$

We may express this differently Put

$$\eta = \frac{\xi''}{\xi'} = -2\frac{\zeta'}{\zeta},$$

so that the *Riccati* equation for η becomes a linear differential equation of the second order,

$$\text{(73)} \qquad \frac{d^2\zeta}{dx^2} + \frac{3}{n+1}P_2\zeta = 0$$

for ζ. Let ζ_1 be any solution of this equation. Then, we may take

$$\xi'(x) = \frac{1}{\zeta_1^2}, \quad \lambda = \zeta_1^{n-1},$$

so that the reduction of (1) to its canonical form may also be accomplished by taking any solution ζ_1 of (73) and putting

$$\text{(74)} \qquad y = \zeta_1^{n-1}\bar{y}, \quad \bar{x} = \int \frac{dx}{\zeta_1^2}.$$

But, the expression for $\bar{x}$ may be written

(74a) $$\bar{x} = \frac{\zeta_2}{\zeta_1},$$

where ζ_2 is another solution of (73) such that $\frac{\zeta_2}{\zeta_1}$ is not a constant. In fact, if ζ_1 and ζ_2 are two solutions of (73) we find at once

$$\zeta_1 \frac{d^2\zeta_2}{dx^2} - \zeta_2 \frac{d^2\zeta_1}{dx^2} = 0,$$

whence

$$\zeta_1 \frac{d\zeta_2}{dx} - \zeta_2 \frac{d\zeta_1}{dx} = const.$$

If we choose the constant equal to unity, and divide by ζ_1^2, integration gives

$$\frac{\zeta_2}{\zeta_1} = \int \frac{dx}{\zeta_1^2}.$$

Moreover this quotient is not a constant, since $\frac{1}{\zeta_1^2}$ is not zero

If the invariant of (1) are all zero, the canonical form of (1) is

$$\frac{d^n \bar{y}}{d\bar{x}^n} = 0,$$

whose general solution is

$$y = a_0 + a_1 x + a_2 \bar{x}^2 + \cdots + a_{n-1} \bar{x}^{n-1}.$$

But this shows that *the general solution of the original equation is*

$$y = a_0 \zeta_1^{n-1} + a_1 \zeta_1^{n-2} \zeta_2 + \cdots + a_{n-1} \zeta_2^{n-1},$$

i. e. *a binary form of the $n - 1^{\text{th}}$ order formed from the solutions of a linear differential equation of the second order.* This is the theorem of *Brioschi* quoted at the end of the last paragraph.

§ 6. Geometrical interpretation.

Among the Ancients, Mathematics was divided into two distinct parts, geometry and arithmetic. Not until the time of *Descartes* and *Fermat* were the two streams which had run along separate channels, united into one. It was then recognized, that geometrical problems could be converted into problems of algebra, while on the other hand a great class of algebraic problems was capable of geometric interpretation. The transformation of an algebraic into a geometric problem, and vice-versa, was accomplished by the introduction of an element, foreign to the problem itself, viz. the system of coordinates. The points of space were put into one to one correspondence with a system of three numbers, their coordinates. To a surface was found to correspond an equation between these three numbers, etc.... From the time of *Descartes* on, the advances of geometry and analysis have been closely connected. Every fundamental notion in one field has

found its important interpretation in the other. Tangent and area of a curve were closely connected with the ideas of derivative and integral of a function, etc. .. Examples of this are sufficiently familiar.

But, as we have noticed already, in any problem of geometry the system of coordinates is really a foreign and arbitrary element. The geometrical relations which we wish to investigate, have nothing to do with this foreign element, which nevertheless makes its appearance in the corresponding equations. If we wish to present a geometrical theory, in an analytical form which shall be perfectly satisfactory, it therefore becomes necessary to write the equations in such a way as to make it evident that they are independent of the particular system of coordinates chosen.

We do this by expressing our equations in an invariant form. *No investigation of analytical geometry can, therefore, be considered satisfactory, unless it has been put into invariant form.*

But why should this valuable aid of a geometrical image be confined to the case of a system of one, two, or three variables? In his *„Ausdehnungslehre"* of 1844, *Grassmann* introduced the idea of geometry in n dimensional space, a point of such a space being determined by n coordinates. On the other hand, the notion of duality in the ordinary three dimensional geometry led to the consideration of other configurations besides points as fundamental elements of space. The principle of duality had shown that if the plane be adopted as element, instead of the point, a new theorem could easily be deduced from any theorem of point geometry. But this new geometry which took planes, as its elements, was three-dimensional as well as the usual point geometry. In 1846 however, *Plücker* took a long step in advance, by taking as fundamental element of space the straight line. Since a straight line is determined by four coordinates, this *line-geometry* of *Plucker's*, of which we shall have to give some account later, is four-dimensional. It now became clear at once, that, by choosing the element of space in an appropriate fashion, a geometry of any number of dimensions could be constructed in ordinary space, or even in the plane. If, therefore, the abstract notion of n-dimensional space should seem to some not to be of a character to assist our imagination, we must remember that an adequate image of this space may be constructed in the ordinary space of experience

We now proceed to explain a few of the notions employed in this theory of higher spaces. Just, as in plane and solid geometry we may introduce *homogeneous* coordinates. We shall say that n quantities $(x_1, \ldots x_n)$ are the homogeneous coordinates of a point P in a space or manifold M_{n-1} of $n-1$ dimensions, if the points of this space can be put into one to one correspondence with the $n-1$ ratios

$$x_1 : x_2 : \cdots : x_n.$$

Let $(x_1^{(1)}, \ldots x_n^{(1)})$ and $(x_1^{(2)}, \ldots x_n^{(2)})$ be the coordinates of two points of M_{n-1}. Then, the quantities

$$x_k = \lambda_1 x_k^{(1)} + \lambda_2 x_k^{(2)}, \quad (k = 1, 2, \ldots n)$$

will be the coordinates of a single infinity of points of M_{n-1}, which we shall denote by M_1. We may speak of this assemblage of points as a straight line. The ratio of $\lambda_1 : \lambda_2$ acquires all values as the point (x_k) moves along the line. Clearly such a line may also be defined by $n - 2$ independent homogeneous linear equations between $x_1, \ldots x_n$.

Let $(x_1^{(3)}, \ldots x_n^{(3)})$ be a third point, which is not on the line M_1. Then, the quantities

$$x_k = \lambda_1 x_k^{(1)} + \lambda_2 x_k^{(2)} + \lambda_3 x_k^{(3)}, \quad (k = 1, 2, \ldots n),$$

will be the coordinates of a double infinity of points of M_{n-1}, which we shall denote by M_2, and whose totality may be called a plane. A plane may also be defined by $n - 3$ independent homogeneous linear equations between $x_1, \ldots x_n$.

In general, m points $(m < n)$ determine in this way a manifold of $m - 1$ dimensions M_{m-1}, provided that they do not all lie in a manifold of fewer than $m - 1$ dimensions. Such a plane manifold M_{m-1}, of $m - 1$ dimensions, may also be defined by $n - m$ independent, homogeneous, linear equations between $x_1, \ldots x_n$.

If we wish to change the fundamental element of our abstract geometry by taking as its fundamental conception not the point, but the manifold M_{m-1}, we must first of all learn how to determine M_{m-1} by coordinates. This we may do, with complete generality as follows. Let us consider m points of M_{m-1} which are not included in any plane manifold of fewer dimensions, and let the matrix of their coordinates be:

$$(\mathfrak{M}) \qquad \begin{matrix} x_1^{(1)}, & x_2^{(1)}, & \ldots & x_n^{(1)}, \\ x_1^{(2)}, & x_2^{(2)}, & \ldots & x_n^{(2)}, \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{(m)}, & x_2^{(m)}, & \ldots & x_n^{(m)}. \end{matrix}$$

From this matrix can be formed

$$N = \frac{n(n-1)\ldots(n-m+1)}{m!}$$

different determinants of the m^{th} order. We define, with *Grassmann*, as homogeneous coordinates of M_{m-1}, N quantities proportional to these determinants. It is not difficult to see that, if we had taken m other points of M_{m-1}, we would have obtained the same coordinates. For, if $y_1^{(i)}, \ldots y_n^{(i)}$, $(i = 1, 2, \ldots m)$ are the coordinates of these other m points, we must have

$$y_k^{(i)} = c_{i1} x_k^{(1)} + c_{i2} x_k^{(2)} + \cdots + c_{im} x_k^{(m)},$$

since they are also points of M_{m-1}. The determinants of the y's would therefore differ from the corresponding determinants of the x's only by the factor, common to all

$$|c_{ik}|, \quad (i, k = 1, 2, \ldots m).$$

This factor, moreover, is not zero. For, if it were, the m new points would be included in a plane manifold of less than $m-1$ dimensions.

We are justified, therefore, in speaking of these N quantities as the homogeneous coordinates of M_{m-1}. For, to every M_{m-1} corresponds one and only one set of their ratios[1]), i. e. the configuration M_{m-1} and its coordinates have been put into one-to-one correspondence. It should not be forgotten, however, that these coordinates are not, in general, independent of each other. In fact, the determinants of the matrix ($\mathfrak{M}$) satisfy certain relations, which we do not, however, need to develop for our present purpose. A single example, of special importance to us, may suffice. Let $(x_1, \ldots x_4)$ and $(y_1, \ldots y_4)$ be two points of ordinary space. In accordance with our general definition, the determinants of the second order, in the matrix,

$$\begin{matrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \end{matrix}$$

will be proportional to the homogeneous coordinates of the line joining the two points. In this case $N = 6$. Put

$$\omega_{ik} = x_i y_k - x_k y_i,$$

and let λ be a proportionality factor. Then we may take

$$\lambda\omega_{12}, \quad \lambda\omega_{13}, \quad \lambda\omega_{14}, \quad \lambda\omega_{23}, \quad \lambda\omega_{42}, \quad \lambda\omega_{34}$$

as the homogeneous coordinates of the line. But, the determinant

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \end{vmatrix}$$

is obviously zero. Upon developing it we find

$$\omega_{12}\omega_{34} + \omega_{13}\omega_{42} + \omega_{14}\omega_{23} = 0,$$

a homogeneous quadratic relation between the six homogeneous line-coordinates.

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1) And, vice-versa, as may be easily shown

We may now apply these notions to our linear differential equation (1). The general theory of such equations establishes the following theorem.

Let $p_1, p_2, \ldots p_n$ be functions of x, regular in the vicinity of $x = a$, i. e. developable in series proceeding according to positive integral powers of $x - a$. Then there exists a system of n regular functions $y_1, y_2, \ldots y_n$, between which there is no relation of the form

$$c_1 y_1 + c_2 y_2 + \cdots c_n y_n = 0$$

with constant coefficients c_k, and each of which satisfies the differential equation. The most general solution of the differential equation, regular in the vicinity of $x = a$, can then be expressed in terms of this fundamental system, in the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where the coefficients c_k are arbitrary constants.

If the equation (1) be integrated we shall therefore have $y_1, \ldots y_n$ expressed as functions of x. We may interpret $y_1, \ldots y_n$ as the homogeneous coordinates of a point P_y in a space M_{n-1} of $n-1$ dimensions. As x changes, P_y moves along a certain curve, C_y, *the integral curve of* (1). Moreover, this curve is not contained in any plane manifold of less than $n-1$ dimensions, since $y_1, \ldots y_n$ satisfy no homogeneous linear equations with constant coefficients. *C_y is therefore a curve of $n-2$ ple curvature.*

But the curve C_y is not determined uniquely by the differential equation. If we put

$$(75) \qquad \bar{y}_k = c_{k1} y_1 + \cdots + c_{kn} y_n, \quad (k = 1, 2, \ldots n),$$

where the determinant

$$|c_{ki}|$$

is different from zero, while the constants c_{ki} are arbitrary, $\bar{y}_1, \ldots \bar{y}_n$ will also form a fundamental system of solutions of (1). We may regard (75) as a transformation of the curve C_y into another $C_{\bar{y}}$. Moreover such transformations shall be called projective transformations of $n-1$ dimensional space. They can be defined geometrically, in a manner altogether analogous to the definition of projective transformations in ordinary space.

We may say, therefore, that the differential equation defines a class of projectively equivalent curves of $n-1$ dimensional space.

If, on the other hand, n linearly independent functions $y_1, \ldots y_n$ of x are given, we can always find an equation of form (1) for which they form a fundamental system. In fact, from the equations

$$y_k^{(n)} + \binom{n}{1} p_1 y_k^{(n-1)} + \cdots + p_n y_k = 0, \quad (k = 1, 2, \ldots n),$$

which express the fact that $y_1, \ldots y_n$ are solutions of (1), $p_1, p_2, \ldots p_n$ can be easily computed. For, the determinant

$$\Delta = \begin{vmatrix} y_1^{(n-1)}, & y_1^{(n-2)}, & \ldots & y_1 \\ y_2^{(n-2)}, & y_2^{(n-2)}, & \ldots & y_2 \\ \cdot & \cdot & \cdot & \cdot \\ y_n^{(n-1)}, & y_n^{(n-2)}, & \ldots & y_n \end{vmatrix}$$

is not identically zero, if the system of n functions $y_1, \ldots y_n$ is linearly independent, i. e. if there is no relation of the form

$$c_1 y_1 + \cdots + c_n y_n = 0$$

with constant, non-evanescent coefficients.[1]) We shall find

$$\binom{n}{k} p_k = -\frac{\Delta_k}{\Delta} \quad (k = 1, 2, \ldots n),$$

where Δ_k denotes the determinant which is obtained from Δ, by substituting for the quantities $y_1^{(n-k)}, \ldots y_n^{(n-k)}$, in it the quantities $y_1^{(n)}, \ldots y_n^{(n)}$.

These expressions show explicitly, that the coefficients of the differential equation (1) are not changed by any projective transformation (75) of the space M_{n-1}. In other words, *the coefficients $p_1, \ldots p_n$ have the same significance for any curve obtained from C_y by projective transformation, as for C_y itself.*

But, if we have $y_1, \ldots y_n$ expressed as functions of x, these expressions still contains arbitrary elements, elements which are, so to speak, accidental and do not belong to the curve itself. In the first place, only the ratios $y_1 : y_2 : \cdots : y_n$ are of importance, since the coordinates are homogeneous. A transformation of the form

$$y = \lambda(x)\bar{y},$$

will therefore not change the curve C_y, since it leaves these ratios unaffected. Further, in place of the parameter x we may take any other

$$\bar{x} = f(x).$$

If, therefore, we form combinations of $p_1, \ldots p_n$ which are not changed by such transformations, they will be expressions which have a significance for the curve itself, independent of the special method of representation, a significance moreover, which is not disturbed by any projective transformation. In other words:

1) We have already quoted this theorem. Cf. *Forsyth,* Treatise on Differential Equations, § 73.

The invariants of a linear homogeneous differential equation characterize the projective properties of its integral curve.

In this connection, the theorem that the coefficients of the semi-canonical form can be expressed in terms of the linear invariants and of their derivatives becomes of fundamental importance. Let $\Theta_3, \Theta_4, \dots \Theta_n$ be given as functions of x. Let, at least one of these invariants, be different from zero, and let Θ_m be the first which does not vanish. Then, by (54), P_2 can be expressed uniquely in terms of Θ_m and $\Theta_{m,1}$. From the expressions of the other invariants, whether they be zero or not, P_3, P_4, etc. ... P_n can be expressed in terms of $\Theta_3, \dots \Theta_n$ and $\Theta_{m,1}$. The semi-canonical form is therefore uniquely determined, i. e. these invariants determine completely a class of projectively equivalent curves. If, on the other hand, all of the linear invariants Θ_m are zero, they also determine a class of projectively equivalent curves. For, we have seen in the last paragraph that the differential equation can then be transformed into

$$\frac{d^n y}{dx^n} = 0,$$

whose solutions

$$y_1 = 1, \ y_2 = x, \dots y_n = x^{n-1}$$

form a fundamental system. This special curve, *the so-called rational normal curve of the $n-1^{\text{th}}$ order* is of special importance in the geometry of $n-1$ dimensional space.

We have found the following fundamental theorem.

Let the linear invariants $\Theta_3, \dots \Theta_m$ be given as arbitrary functions of x. Let Θ_m be the first of these which does not vanish identically, and let the corresponding quadri-derivative $\Theta_{m,1}$ be also given as an arbitrary function of x. These $n-1$ functions determine a curve of $n-1$ dimensional space uniquely except for projective transformations.

This theorem corresponds precisely to the fundamental theorem of the metrical theory of surfaces in ordinary space, which asserts that a surface is determined, except for its position in space, by the coefficients of its fundamental quadratic forms. In that case, however, the formulation of the theorem is far less simple, because the coefficients of these quadratic forms are not independent of each other, but must satisfy certain relations.

Let the functions $y_1, \dots y_n$ be regular for $x = a$, so that for sufficiently small values of $x = a$, we may express them in the form

$$y_k = y_{k0} + y'_{k0}(x-a) + \frac{1}{2} y''_{k0}(x-a)^2 + \cdots,$$

where $y_{k0}, y'_{k0}, y''_{k0}$, etc. denote the values of y_k, y_k', y_k'', etc. for $x = a$. Put $x - a = h$. Join the point P which corresponds to $x = a$, to the point Q, which corresponds to $x = a + h$, by a straight line.

In accordance with our definition of a straight line, the coordinates of any point on this line may be written in the form

$$\lambda_1 y_{k0} + \lambda_2 y_k.$$

Therefore, the point whose coordinates are

$$\frac{y_k - y_{k0}}{h},$$

is a point of this line. The limiting position of the line, as h approaches zero, is called the tangent of C_y at P_y. We see, therefore, that the quantities

$$\lim_{h-0} \frac{y_k - y_{k0}}{h} = (y_k')_{x=a}, \quad (k = 1, 2, \ldots n)$$

represent the coordinates of a point on the tangent. This point will, moreover, in general, be different from the point P_y itself. For, else, we would have for all values of x

$$y_k' = \lambda y_k, \quad (k = 1, 2, \ldots n),$$

i. e. $y_1, \ldots y_n$ would all satisfy the same linear differential equation of the first order. They could, therefore, differ from each other only by constant factors, i. e they could not form a fundamental system of (1).

In the same way we may define the osculating plane, and show that three of its points, not in general collinear, are

$$y_k; \quad y_k'; \quad y_k''; \quad (k = 1, 2, \ldots n).$$

In general we may take m points upon the curve; P_y, and $m - 1$ others. They determine a plane manifold M_{m-1} of $m - 1$ dimensions. We allow all of these points to approach P_y as a limit. The resulting limiting plane manifold M_{m-1} shall be spoken of as osculating the curve C_y at P_y. The m points whose coordinates are given by

$$y_k;\ y_k';\ y_k'';\ \ldots y_k^{(m-1)}; \quad (k = 1, 2, \ldots n),$$

are in general m distinct points of M_{m-1}, which are not included in a plane manifold of less than $m - 1$ dimensions. The coordinates of any point of M_{m-1} may then be written in the form

$$x_k = \lambda_0 y_k + \lambda_1 y_k' + \cdots + \lambda_{m-1} y_k^{(m-1)}, \quad (k = 1, 2, \ldots n).$$

Let us consider, in particular, the osculating plane manifold of $n - 2$ dimensions, M_{n-2}. In accordance with our general definition, its coordinates are n in number, and may be taken proportional to the minors of $y_1^{(n-1)}, y_2^{(n-1)}, \ldots y_n^{(n-1)}$ in the determinant Δ. But

according to (68), the solutions of the *Lagrange adjoint* of (1) are also proportional to these same minors of Δ. We may, therefore, say that the solutions of (1) and of its *Lagrange adjoint* correspond to each other by the principle of duality. That this is true for $n=3$ and $n=4$ is evident. That it is true in general will become clear if we formulate the principle of duality for space of $n-1$ dimensions. We have seen that $n-1$ points determine an M_{n-2}. But since M_{n-2} can also be defined as the locus of points which satisfy one linear, homogeneous equation between $x_1, \ldots x_n$, it is clear that $n-1$ plane manifolds M_{n-2} will in general determine a point as their one common element, its coordinates being the one solution of $n-1$ homogeneous linear equations. Similarly, $n-k$ points determine an M_{n-k-1}, and this can also be determined by k equations, i. e. as the locus of points common to k manifolds M_{n-2}. Therefore, in the same way $n-k$ manifolds M_{n-2} have in common a manifold M_{k-1} Consider any theorem in the geometry of M_{n-1} which is concerned only with intersections of plane manifolds. We shall be able to deduce from it another theorem, by putting everywhere for the word *point*, the word *plane manifold of $n-2$ dimensions*, for the word *plane manifold of $k-1$ dimensions*, the word *plane manifold of $n-k-1$ dimensions.*

If n is even, there is a self-dual plane manifold $M_{\frac{n}{2}}$. Thus in three dimensional geometry, straight lines are the self-dual elements.

We have assumed that $y_1, \ldots y_n$ were point coordinates. The curve C_y has been defined as the locus of a moving point. We have seen, further, that the coordinates of the osculating M_{n-2} may be identified with $z_1, \ldots z_n$ the solutions of the *Lagrange* adjoint equation. We may therefore regard this latter equation as having the same integral curve as (1), the curve however not being regarded as the locus of a point but as the envelope of its osculating plane manifolds of $n-2$ dimensions. We may also, however, interpret $z_1, \ldots z_n$ as point coordinates. Then, the integral curve C_z is in general different from C_y. Its points will satisfy the same equations which are satisfied by the osculating plane manifolds of $n-2$ dimensions of C_y. The curves C_y and C_z are therefore dualistic transformations of each other.

We have seen in the last paragraph how the invariants of the two equations were related to each other. Our fundamental theorem may therefore be completed as follows:

If the linear invariants of even weight have the same values, and if those of odd weight have opposite values for two curves of the space M_{n-1}, these curves are dualistic to each other.

If the linear invariants of odd weight are zero, the curve is self-dual.

The dualistic character of the correspondence of the two curves C'_y and C_z may also be seen by setting up the theory of the polars with respect to the quadric

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0.$$

The relations (68a) show that $z_1, \dots z_n$ is the pole of the M_{n-2} osculating C'_y at P_y with respect to this quadric.

§ 7. The relation of the invariants of a linear differential equation to Halphen's differential invariants.

We have seen that $p_1, \dots p_n$ can be expressed in terms of $y_1, \dots y_n$, and that these expressions are unaffected by projective transformations. The invariants, for the sake of simplicity let us consider *absolute* invariants, can only be functions of the ratios $y_1 : y_2 : \cdots : y_n$ since they are not changed by any transformation of the form $\bar{y} = \lambda(x) y$. They are also left unchanged by any transformation of the independent variable. We may therefore find a special form for the expression of these invariants, by introducing the quotients

$$Y_1 = \frac{y_2}{y_1}, \; Y_2 = \frac{y_3}{y_1}, \dots Y_{n-1} = \frac{y_n}{y_1},$$

and taking Y_1 as independent variable. The invariants will then become such functions of $Y_1, Y_2, \dots Y_{n-1}$, and of the derivatives of these quantities with respect to Y_1, as are left unchanged by any projective transformation. These functions are *Halphen's differential invariants*. *Halphen* has worked out their theory for $n = 3$ and for $n = 4$, and it is upon this basis that he has constructed his theory of plane and space curves. We must remember however that he did not obtain these invariants in this way. His method, applied to the general case would be as follows. Let $Y_1, \dots Y_{n-1}$ be the (unhomogeneous) coordinates of a point in the space M_{n-1}. Let

$$Y_k = f_k(Y_1), \quad (k = 2, 3, \dots n-1)$$

be the equations of a curve in this space. Then it becomes *Halphen*'s problem to find functions of $Y_1, Y_2, \dots Y_n, \frac{dY_k}{dY_1}, \frac{d^2 Y_k}{dY_1^2}$, etc. which remain invariant for all transformations of the form

$$\bar{Y}_k = \frac{c_{k0} + c_{k1} Y_1 + \cdots + c_{k,n-1} Y_{n-1}}{c_0 + c_1 Y_1 + \cdots + c_{n-1} Y_{n-1}}, \quad (k = 1, 2, \dots n-1),$$

where the coefficients c_k and c_{ik} are constants.

This unsymmetrical and unhomogeneous formulation of the problem is manifestly a disadvantage. It is easy enough to obtain the un-

homogeneous form when the homogeneous is known, but the inverse process is far more difficult. We shall, therefore, make little use of *Halphen*'s differential invariants, but shall deduce *Halphen*'s theorems on plane and space curves directly from the differential equation. Doubtlessly, *Halphen* would also have done this, if he had noticed this connection at the time. In his later papers, in which he makes use of the differential equation and its invariants, his point of view is no longer geometrical, except in a secondary way. But even then the form, into which we shall put this theory in the following chapters, could scarcely have occurred to *Halphen*. For, we shall find that the geometrical theory of the semi-covariants is essential for this purpose, and, at least for $n = 4$, this theory requires as prerequisite a general projective theory of ruled surfaces. But *Halphen*, never mentions these semi-covariants, and the general theory of ruled surfaces is of more recent date.

Examples.

Ex. 1. Compute the expression $A_{m,m-3}$ defined by equation (15).

Ex. 2. Show that in general

$$A_{mk} = \lim_{\varrho \to 0} \frac{d^m}{d\varrho^m} [\varphi(x+\varrho) - \varphi(x)]^k. \qquad \text{(Schlömilch.)}$$

Ex. 3. Making use of the result of Ex. 1, find the general expression for P_3.

Ex. 4. Denote the Schwarzian derivative by $\{\xi, x\}$. Prove that it vanishes if $\xi = \frac{ax+b}{cx+d}$, a, b, c, d being constants. Prove the following formulae:

$$\{\xi, x\} = -\left(\frac{d\xi}{dx}\right)^2 \{x, \xi\}; \quad \left\{\frac{a\xi+b}{c\xi+d}, x\right\} = \{\xi, x\};$$

$$\{\xi, x\} = \left(\frac{dy}{dx}\right)^2 [\{\xi, y\} - \{x, y\}];$$

$$\{\xi, x\} = \left(\frac{dy}{dx}\right)^2 \{\xi, y\} - \left(\frac{dv}{dx}\right)^2 \{x, v\} + \left(\frac{dv}{dx}\right)^2 \{y, v\}. \quad \text{(Cayley.)}$$

Ex. 5. Compute the invariants and covariants in their canonical form for $n = 3$ and for $n = 4$.

Ex. 6. Compute the invariants and covariants of the equation

$$y^{(3)} + \frac{a}{x^2} y' + \frac{b}{x^3} y = 0.$$

Ex. 7. Find the adjoints of the general equations of the third and fourth order.

Ex. 8. Reduce the following equations to their semi-canonical form, and then solve

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=c^2y;\quad \frac{d^2y}{dx^2}-\frac{3x+1}{x^2-1}\frac{dy}{dx}+y\left\{\frac{6(x+1)}{(x-1)(3x+5)}\right\}^2=0;$$

$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=c^2y;\quad \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^2}{x^4}y=0;$$

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0;\ (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=0\quad \text{(Forsyth.)}$$

Ex. 9 If s is the quotient of any two solutions of

$$\frac{d^2y}{dx^2}+Iy=0,$$

then s satisfies the equation

$$\{s, x\}=2I. \qquad \text{(Kummer, Schwarz.)}$$

Ex. 10. Find the linear differential equations of the third order whose fundamental solutions are

$$1,\ x,\ x^2;\quad x+\sqrt{x},\ x-\sqrt{x},\ x^2;\quad \sin kx,\ \cos kx,\ x.$$

CHAPTER III.

PROJECTIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES.

§ 1. The invariants and covariants for $n=3$.

The geometrical interpretation developed in outline, in Chapter II, shows that the general projective theory of plane curves may be attached to the discussion of the linear differential equation of the third order

(1) $$y^{(3)}+3p_1y''+3p_2y'+p_3y=0.\text{ [1]}$$

The two seminvariants [Chapter II, (9)] are

(2) $$P_2=p_2-p_1^2-p_1',\quad P_3=p_3-3p_1p_2+2p_1^3-p_1''.$$

For the semi-covariants it seems desirable to change the notation. We shall denote them by z and ϱ, so that

(3) $$z=y'+p_1y,\quad \varrho=y''+2p_1y'+p_2y.$$

We have deduced, in the general case, the canonical form of the invariants. For our more detailed discussion of the case $n=3$ we shall need also their uncanonical form. Moreover it will be necessary to have at hand explicitly the formulae, which express the effect upon the coefficients of (1) of the transformation

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1) Of course the equation (1) may be interpreted dualistically as the equation of a cone.

$$(4) \qquad \bar{x} = \xi(x)$$

of the independent variable We find, either directly, or by specializing the general equations (21),

$$(5) \quad \bar{p}_1 = \frac{1}{\xi'}(p_1 + \eta), \; \bar{p}_2 = \frac{1}{(\xi')^2}\left[p_2 + \eta p_1 + \frac{1}{3}\mu + \frac{1}{2}\eta^2\right], \; \bar{p}_3 = \frac{1}{(\xi')^3}p_3,$$

where

$$(6) \qquad \xi' = \frac{d\xi}{dx}, \text{ etc.,} \quad \eta = \frac{\xi''}{\xi'}, \quad \mu = \eta' - \frac{1}{2}\eta^2.$$

Consequently, we find

$$(7) \qquad \begin{aligned} \bar{z} &= \frac{1}{\xi'}(z + \eta y), \\ \bar{\varrho} &= \frac{1}{(\xi')^2}\left[\varrho + \eta z + \left(\frac{1}{3}\mu + \frac{1}{2}\eta^2\right)y\right], \end{aligned}$$

so that $\frac{z}{y}$ and $\frac{\varrho}{y}$ are transformed cogrediently with p_1 and p_2.

If we denote $\frac{d\bar{p}_k}{d\bar{x}}$ by $\bar{p}_k'$, $\frac{d^2\bar{p}_k}{d\bar{x}^2}$ by $\bar{p}_k''$, etc., we find

$$\begin{aligned} \bar{p}_1' &= \frac{1}{(\xi')^2}\left[p_1' - \eta p_1 + \mu - \frac{1}{2}\eta^2\right], \\ \bar{p}_1'' &= \frac{1}{(\xi')^3}\left[p_1'' - 3\eta p_1' - \left(\mu - \frac{3}{2}\eta^2\right)p_1 + \mu' - 3\eta\mu + \frac{1}{2}\eta^3\right], \end{aligned}$$

whence

$$(8) \qquad \begin{aligned} \bar{P}_2 &= \frac{1}{(\xi')^2}\left[P_2 - \frac{2}{3}\mu\right], \\ \bar{P}_3 &= \frac{1}{(\xi')^3}\left[P_3 - 3\eta P_2 - \mu' + 2\mu\eta\right]. \end{aligned}$$

The former of these equations follows at once, if we put $n = 3$ in equation (27) of Chapter II. We find further

$$\bar{P}_2' = \frac{1}{(\xi')^3}\left[P_2' - 2\eta P_2 - \frac{2}{3}\mu' + \frac{4}{3}\mu\eta\right],$$

so that

$$\bar{\Theta}_3 = \frac{1}{(\xi')^3}\Theta_3,$$

if

$$(9) \qquad \Theta_3 = P_3 - \frac{3}{2}P_2'.$$

Θ_3 is the one linear invariant which exists in this case. For the canonical form it reduces, of course, to P_3. The quadri-derivative $\Theta_{3\cdot 1}$ of Θ_3 we shall denote in this case by Θ_8, since there is no danger of confusion with any other invariant of weight 8. We have, according to equation (54) of Chapter II,

$$(10) \qquad \Theta_8 = 6\Theta_3\Theta_3'' - 7(\Theta_3')^2 - 27P_2\Theta_3^2.$$

The invariance of Θ_8 may also be tested directly. We know by the general theorem that all invariants are functions of Θ_3, Θ_8 and of their successive Jacobians. Of these we shall need

$$(11)\qquad \begin{aligned} \Theta_{12} &= 3\Theta_3\Theta_8' - 8\Theta_8\Theta_3', \\ \Theta_{16} &= \Theta_3\Theta_{12}' - 4\Theta_{12}\Theta_3', \\ \Theta_{21} &= 2\Theta_8\Theta_{12}' - 3\Theta_{12}\Theta_8', \end{aligned}$$

between which there is the relation

$$(12)\qquad \Theta_{12}^2 + \Theta_3\Theta_{21} - 2\Theta_8\Theta_{16} = 0.$$

The functions

$$(13)\qquad \begin{aligned} C_2 &= z^2 - 2y\varrho - P_2 y^2, \\ C_4 &= \Theta_3' y + 3\Theta_3 z, \end{aligned}$$

are covariants of weight 2 and 4 respectively. All other covariants may be expressed as functions of these and of invariants. We shall later find another covariant, capable of a simple geometrical interpretation, to replace C_2.

In our special case, we have to interpret y_1, y_2, y_3, the members of a fundamental system of (1), as homogeneous coordinates of a point P_y in a plane. As x varies, P_y describes a plane curve C_y. If we denote by u_1, u_2, u_3 the coordinates of the tangent to C_y at P_y, they form a fundamental system for the *Lagrange adjoint* of (1), which is in this case [cf. equation (62) of Chapter II],

$$(14)\qquad u^{(3)} - 3p_1 u'' + 3(p_2 - 2p_1')u' - (p_3 - 3p_2' + 3p_1'')u = 0$$

Its seminvariants are

$$\Pi_2 = P_2, \quad \Pi_3 = -P_3 + 3P_2',$$

so that its invariants differ from those of (1) only in having $-\Theta_3$ in place of Θ_3, in accordance with the general theory.

The *Laguerre-Forsyth* canonical form of (1) will be obtained by making the transformation

$$\bar{y} = \lambda(x)y, \quad \bar{x} = \xi(x),$$

where $\xi(x)$ is chosen so as to satisfy the equation

$$P_2 - \frac{2}{3}\mu = 0,$$

which reduces P_2 to zero, and where

$$\lambda = (\xi')^{-1},$$

in accordance with the general theory, so that $\bar{p}_1 = \bar{p}_2 = 0$. The equation assumes the form

$$\bar{y}^{(3)} + \bar{P}_3\bar{y} = 0,$$

for which $\bar{\Theta}_3 = \bar{P}_3$.

Since

$$\bar{\Theta}_3 = \frac{1}{(\xi')^3}\Theta_3,$$

if we put

$$\bar{x} = \int \sqrt[3]{\Theta_3}\, dx,$$

$\bar{\Theta}_3$ will be equal to unity. We may then choose $\lambda(x)$ so as to have $\bar{p}_1 = 0$. We shall speak of the canonical form which is characterized by the conditions

$$\Theta_3 = 1, \quad p_1 = 0,$$

as the *Halphen* canonical form Equation (1) may always be reduced to the *Halphen* canonical form if Θ_3 does not vanish identically.

If $\Theta_3 = 0$, the *Laguerre-Forsyth* canonical form becomes

$$\bar{y}^{(3)} = 0,$$

so that C_y is a conic. If $\Theta_3 \neq 0$, let Θ_3 and Θ_8 be given as functions of x. We can solve (10) and (9) for P_2 and P_3. We find very easily therefore, the special case of our general fundamental theorem. *The invariants Θ_3 and Θ_8 determine a plane curve except for projective transformations. If for all pairs of corresponding points of two curves C_y and C'_y,*

$$\Theta'_3 = -\Theta_3, \quad \Theta'_8 = \Theta_8,$$

the two curves are dualistic to each other.

Let us call a curve *identically self-dual*, if a dualistic transformation exists which converts it into itself *point for point*, so that by this transformation, every point of the curve is converted into the tangent at that point, and every tangent into its point of contact. Then we can say, that *the only identically self-dual plane curves are the conics.* For, Θ_3 must vanish for such curves

§ 2. The equations of the osculating conic and cubic.

Put $y = y_1, y_2, y_3$ in the expressions (4). We find in this way two other points P_z and P_ϱ, which describe two curves C_z and C_ϱ, semi-covariantly connected with C_y. P_z is clearly a point on the tangent to C_y at P_y, while P_ϱ is some other point of the plane. If we assume that P_y is not a point of inflection, the three points P_y, P_z, P_ϱ will not be collinear. We may, therefore, take these points as vertices of the triangle of reference. Moreover, we may choose the unit point of our system of homogeneous coordinates in such a way that an expression of the form

$$x_1 y + x_2 z + x_3 \varrho$$

shall represent the point whose coordinates are precisely x_1, x_2, x_3. The geometrical significance of this triangle of reference, which we have only defined analytically, will appear later, as a consequence of the developments which we proceed to make.

Let the differential equation be written in the *Laguerre-Forsyth* canonical form,

$$y^{(3)} + P_3 y = 0,$$

so that $p_1 = p_2 = 0$, $p_3 = P_3$. Let $x = a$ be an ordinary point for the function P_3 of x, so that the members of a fundamental system may be expressed as series proceeding according to positive integral powers of $x - a$, convergent for values of $|x - a|$ sufficiently small. For greater convenience in writing we shall put $a = 0$. In fact we may, by the transformation

$$x - a = \bar{x}$$

always reduce the developments to this form. We proceed, therefore, to express the solution Y of our equation as a power series in x, and we shall actually calculate the coefficients up to the ninth order.

We shall find

$$Y = y + y'x + \frac{1}{2} y''x^2 + \cdots + \frac{1}{9!} y^{(9)}x^9 + \cdots,$$

an expression which may be written in the form

$$Y = y_1 y + y_2 z + y_3 \varrho,$$

where y_1, y_2, y_3 are themselves such power-series, which will represent the curve C_y up to terms of the 9^{th} order in the vicinity of the point P_y, referred to the system of coordinates which has just been defined.

Since $p_1 = p_2 = 0$, we find by successive differentiation:

$$\begin{aligned}
&y' = z, \quad y'' = \varrho, \quad y^{(3)} = -P_3 y, \quad y^{(4)} = -P_3' y - P_3 z,\\
&y^{(5)} = -P_3'' y - 2P_3' z - P_3 \varrho,\\
&y^{(6)} = -(P_3^{(3)} - P_3^2) y - 3P_3'' z - 3P_3' \varrho,\\
(15) \quad &y^{(7)} = -(P_3^{(4)} - 5P_3 P_3') y - (4P_3^{(3)} - P_3^2) z - 6P_3'' \varrho,\\
&y^{(8)} = -[P_3^{(5)} - 11P_3 P_3'' - 5(P_3')^2] y + (-5P_3^{(4)} + 7P_3 P_3') z\\
&\qquad - (10P_3^{(3)} - P_3^2) \varrho,\\
&y^{(9)} = -[P_3^{(6)} - 21P_3 P_3^{(3)} - 21P_3' P_3'' + P_3^3] y\\
&\qquad + [-6P_3^{(5)} + 18P_3 P_3'' + 12(P_3'')^2] z - (15P_3^{(4)} - 9P_3 P_3') \varrho.
\end{aligned}$$

If, therefore, we put Y into the form indicated above, we shall have:

$$\begin{aligned}
&y_1 = 1 - \frac{P_3}{3!} x^3 - \frac{P_3'}{4!} x^4 - \frac{P_3''}{5!} x^5 - \frac{P_3^{(3)} - P_3^2}{6!} x^6 - \frac{P_3^{(4)} - 5P_3 P_3'}{7!} x^7\\
&\qquad - \frac{1}{8!} \{P_3^{(5)} - 11P_3 P_3'' - 5(P_3')^2\} x^8\\
&\qquad - \frac{1}{9!} (P_3^{(6)} - 21P_3 P_3^{(3)} - 21P_3' P_3'' + P_3^3) x^9 + \cdots,\\
&y_2 = x - \frac{P_3}{4!} x^4 - \frac{2P_3'}{5!} x^5 - \frac{3P_3''}{6!} x^6 - \frac{4P_3^{(3)} - P_3^2}{7!} x^7\\
(16) \quad &\qquad - \frac{1}{8!} (5P_3^{(4)} - 7P_3 P_3') x^8\\
&\qquad - \frac{1}{9!} \{6P_3^{(5)} - 18P_3 P_3'' - 12(P_3')^2\} x^9 + \cdots,\\
&y_3 = \frac{1}{2} x^2 - \frac{P_3}{5!} x^5 - \frac{3P_3'}{6!} x^6 - \frac{6P_3''}{7!} x^7 - \frac{10P_3^{(3)} - P_3^2}{8!} x^8\\
&\qquad - \frac{15P_3^{(4)} - 9P_3 P_3'}{9!} x^9 + \cdots.
\end{aligned}$$

We shall find therefore:

$$(17)\quad \begin{aligned} y_2^2 - 2y_1y_3 = {} & \frac{1}{10}P_3x^5 + \frac{1}{60}P_3'x^6 + \frac{1}{420}P_3''x^7 \\ & + \frac{1}{40320}(12P_3^{(3)} - 84P_3^2)x^8 \\ & + \frac{1}{362880}(12P_3^{(4)} - 504P_3P_3')x^9 + \cdots \end{aligned}$$

This equation gives an important result. Consider the conic

$$x_2^2 - 2x_1x_3 = 0.$$

To find its intersections with the curve C_y, we substitute into its left member $x_k = y_k$. Equation (17) gives the result of this substitution, and shows that the development of

$$y_2^2 - 2y_1y_3$$

coincides with that of

$$x_2^2 - 2x_1x_3$$

up to and including terms of the 4th order. In other words, this conic has at P_y a contact of the 4th order with C_y, or it has five consecutive points in common with it. It is, therefore, the *osculating conic.*

Put

$$(18)\quad \begin{aligned} \Omega_1(y) &= 5(y_2^2 - 2y_1y_3)(P_3'y_3 - 3P_3y_2) + 12P_3^2y_3^3, \\ \Omega_2(y) &= 5(y_2^2 - 2y_1y_3)(21P_3y_1 - P_3''y_3) - 42P_3^2y_2y_3^2 \\ & \qquad - 14P_3P_3'y_3^3. \end{aligned}$$

We shall find

$$(19)\quad \begin{aligned} \Omega_1(y) = {} & \frac{1}{168}[7(P_3')^2 - 6P_3P_3'']x^8 \\ & + \frac{1}{3360}[20P_3'P_3'' - 15P_3^{(3)}P_3 + 63P_3^3]x^9 + \cdots, \\ \Omega_2(y) = {} & \frac{1}{480}[15P_3^{(3)}P_3 - 20P_3'P_3'' - 567P_3^3]x^8 \\ & + \frac{1}{2016}[7P_3P_3^{(4)} - 12(P_3'')^2 - 882P_3^2P_3']x^9 + \cdots, \end{aligned}$$

so that finally

$$(20)\quad \begin{aligned} & 7(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)\Omega_1(y) + 20[6P_3P_3'' - 7(P_3')^2]\Omega_2(y) \\ & = \frac{x^9}{20\cdot 21\cdot 24}[21(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)(20P_3'P_3'' \\ & \qquad - 15P_3P_3^{(3)} + 63P_3^3) \\ & + 100\{6P_3P_3'' - 7(P_3')^2\}\{7P_3P_3^{(4)} - 12(P_3'')^2 - 882P_3^2P_3'\}] + \cdots. \end{aligned}$$

A plane cubic is determined by nine points. We shall speak of the cubic which has, at P_y, nine consecutive points in common with C_y, i. e. which has with C_y a contact of the eighth order as the *osculating*

cubic. As, in the case of the conic, we find its equation at once from (20). Uniting the two results, we may recapitulate as follows:

Referred to the system of coordinates defined by the semi-covariants, when the differential equation is written in the Laguerre-Forsyth canonical form, the equations of the osculating conic and cubic are respectively

$$x_2^2 - 2x_1x_3 = 0,$$

$$(21) \qquad 7(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)\Omega_1(x) \\ + 20[6P_3P_3'' - 7(P_3')^2]\Omega_2(x) = 0.$$

§ 3. Geometrical interpretation of the semi-covariants.

We have already noticed that P_z is a point of the tangent constructed to C_y at P_y. Moreover (7) shows that a change of the independent variable has the effect of displacing P_z along the tangent. We may even choose the independent variable so, as to make P_z coincide with any point of the tangent. If we mark upon every tangent of C_y a point, the function $\eta(x)$ of equation (7) may be so chosen as to make the curve C_z coincide with the locus of these points. Unless, therefore, the independent variable be chosen in some special way, the curve C_z has no specific relation to C_y. It may serve merely as a geometrical image of the independent variable x. This image does not necessarily change if the independent variable be transformed. For, as (7) shows, since the coordinates employed are homogeneous, two values of $\xi(x)$ which give rise to the same value of $\eta(x)$ transform C_z into the same curve $C_{\bar{z}}$. In other words, a linear transformation

$$\bar{x} = ax + b$$

of the independent variable, where a and b are constants, has no geometrical significance.

We may, therefore, look upon the curve C_z as defining the independent variable of the differential equation, except for such an inessential linear transformation. The curve C_ϱ will then be determined uniquely. It remains, therefore, to find the relation between the points P and P_ϱ.

Let us assume now that $P_2 = 0$, so that the differential equation is in its canonical form. Let us make all transformations of the independent variable, which do not disturb this condition. As (8) shows we must have $\mu = 0$, or

$$\eta' = \frac{1}{2}\eta^2,$$

so that the locus of all points $P_{\bar{\varrho}}$ as given by (7), becomes

$$\bar{\varrho} = \frac{1}{(\xi')^2}\left[\varrho + \eta z + \frac{1}{2}\eta^2 y\right],$$

or

$$x_1 = \frac{1}{2}\eta^2, \quad x_2 = \eta, \quad x_3 = 1,$$

where η may have any numerical value. The elimination of η gives

$$x_2^2 - 2x_1x_3 = 0$$

as the equation of this locus. In other words: *if $P_2 = 0$, the point P_ϱ is upon the osculating conic. If all of the transformations are made, which do not disturb the condition $P_2 = 0$, P_ϱ assumes successively all positions upon the osculating conic.*

We are now in a position to determine the equation of the osculating conic, referred to the triangle of reference $P_yP_zP_\varrho$, even if P_2 is not equal to zero. For, if $P_2 \neq 0$, P_2 will vanish, if η is any solution of the *Riccati* equation

$$\mu = \eta' - \frac{1}{2}\eta^2 = \frac{3}{2}P_2.$$

The points P_ϱ which correspond to all of these solutions, are by our previous result, the points of the osculating conic. We find

$$\bar{\varrho} = \frac{1}{(\xi')^2}\left[\varrho + \eta z + \left(\frac{1}{2}P_2 + \frac{1}{2}\eta^2\right)y\right],$$

so that

$$x_1 = \frac{1}{2}(P_2 + \eta^2), \quad x_2 = \eta, \quad x_3 = 1,$$

are the parametric equations of the conic. Eliminating η, we find

(22) $$x_2^2 - 2x_1x_3 + P_2x_3^2 = 0,$$

the equation of the osculating conic, referred to the triangle of reference $P_yP_zP_\varrho$, when this triangle is not specialized.

The polar of any point (x_1', x_2', x_3') with respect to this conic is

$$-x_3'x_1 + x_2'x_2 + (P_2x_3' - x_1')x_3 = 0,$$

so that the polar of P_z, or $(0, 1, 0)$, is $x_2 = 0$, i. e.: *the line P_yP_ϱ is the polar of P_z with respect to the osculating conic.*

The line P_yP_ϱ, which has now a known geometrical significance, intersects the osculating conic in P_y and in another point P_α, whose coordinates are given by the expression

(23) $$\alpha = P_2y + 2\varrho.$$

As x changes and P_y moves along the curve C_y, the line P_yP_ϱ will envelop a certain curve C_γ. We proceed to determine the point P_β at which P_yP_ϱ touches C_γ. In order to do this, we allow x to increase by δx, where δx is an infinitesimal The line P_yP_ϱ will assume the position $P_{y+y'\delta x}P_{\varrho+\varrho'\delta x}$. As δx approaches zero, the intersection of this latter line with P_yP_ϱ will approach a certain limiting position. This limit will be the point P_β.

We find, by differentiation

$$(24)\qquad \begin{aligned} y' &= -p_1 y + z, \\ \varrho' &= (-P_3 + P_2')y - 2P_2 z - p_1 \varrho, \end{aligned}$$

whence

$$y + y'\delta x = y(1 - p_1\delta x) + z\,\delta x,$$
$$\varrho + \varrho'\delta x = (P_2' - P_3)y\,\delta x - 2P_2 z\,\delta x + \varrho(1 - p_1\delta x)$$

The line joining these points intersects $P_y P_\varrho$ in the point

$$2P_2(y + y'\delta x) + \varrho + \varrho'\delta x = (1 - p_1\delta x)(2P_2 y + \varrho) + (P_2' - P_3)y\,\delta x,$$

whose limit is

$$(25)\qquad \beta = 2P_2 y + \varrho.$$

The cross-ratio of the four points $P_y, P_\alpha, P_\varrho, P_\beta$ is given by

$$(P_\alpha, P_y, P_\beta, P_\varrho) = 4.$$

The point P_ϱ is completely determined by these considerations. *Upon the polar of P_z, with respect to the osculating conic, we mark the points P_y and P_α in which it meets the conic, as well as the point P_β at which it touches its envelope. The point P_ϱ is then determined by the condition that the cross-ratio of the four points shall be equal to 4.*

This construction becomes indeterminate if $P_2 = 0$. In that case however, P_ϱ and P_β coincide with P_α the second intersection of $P_y P_\varrho$ with the osculating conic. In this case, therefore, $P_y P_\varrho$ is a tangent of the curve described by P_ϱ. This gives us the interpretation of the condition $P_2 = 0$, which is characteristic of the *Laguerre-Forsyth* canonical form.

The most general curve $C_{\bar\varrho}$ depends upon an arbitrary function $\eta(x)$. If this function is chosen in a definite manner the curve $C_{\bar z}$ is determined uniquely, and therefore, by the above construction also the curve $C_{\bar\varrho}$. Among these curves $C_{\bar\varrho}$ there exists a single infinity such that their tangents at $P_{\bar\varrho}$ pass through the corresponding point P_y of C_y. These are the special curves $C_{\bar\varrho}$ which are obtained by reducing the differential equation to the Laguerre-Forsyth canonical form. Moreover, if we construct all of the points $P_{\bar\varrho}$, one on each of these ∞^1 curves, which are thus related to a definite point P_y of C_y, their locus is the conic which osculates C_y at P_y. Finally, any four of the curves $C_{\bar z}$, which correspond to four of these special curves C_ϱ, intersect all of the tangents of C_y in point-rows of the same anharmonic ratio.

The last remark results from the fact that the equation

$$\eta' - \frac{1}{2}\eta^2 = \frac{3}{2}P_2,$$

which determines these ∞^1 curves, is of the *Riccati* form. The anharmonic-ratio of any four solutions of such an equation is always constant.[1])

1) See for example *Forsyth*, A treatise on differential equations, p. 190. 3d edition.

§ 4. The eight-pointic cubics, the Halphen point, coincidence points.

Let us again assume that (1) has been written in the *Laguerre-Forsyth* canonical form, so that $p_1 = p_2 = 0$, $p_3 = P_3$. We have seen in § 2, equations (19), that each of the cubic curves

$$(26)\qquad \begin{aligned} \Omega_1(x) &= 5(x_2^2 - 2x_1x_3)(P_3'x_3 - 3P_3x_2) + 12P_3^2x_3^3 = 0, \\ \Omega_2(x) &= 5(x_2^2 - 2x_1x_3)(21P_3x_1 - P_3''x_3) - 42P_3^2x_2x_3^2 \\ &\qquad - 14P_3P_3'x_3^3 = 0, \end{aligned}$$

has eight consecutive points in common with C_y at P_y, or has with C_y at P_y a contact of the seventh order. The same is therefore true of each of the ∞^1 cubics

$$(27)\qquad \alpha\Omega_1(x) + \beta\Omega_2(x) = 0,$$

where α and β are constants. We shall speak of these cubics as the *eight-pointic cubics* of P_y. Among these there is, of course, a nine-pointic cubic, i. e. the osculating cubic of C_y at P_y. We have seen, [cf. equation (21)], that its equation is obtained by putting in (27),

$$(28)\qquad \alpha = 7(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3),\quad \beta = 20[6P_3P_3'' - 7(P_3')^2].$$

But, the eight-pointic cubics have a ninth point in common, which we shall call the *Halphen point* of P_y. We proceed to find its expression.

We find from $\Omega_2 = 0$,

$$x_1 = \frac{5x_2^2(P_3'x_3 - 3P_3x_2) + 12P_3^2x_3^3}{10(P_3'x_3 - 3P_3x_2)x_3}.$$

This gives further

$$5(x_2^2 - 2x_1x_3) = \frac{-12P_3^2x_3^3}{P_3'x_3 - 3P_3x_2},$$

$$21P_3x_1 - P_3''x_3 = \frac{(P_3'x_3 - 3P_3x_2)(105P_3x_2^2 - 10P_3''x_3^2) + 252P_3^3x_3^3}{10(P_3'x_3 - 3P_3x_2)x_3},$$

whence, substituting in $\Omega_2 = 0$,

$$\begin{aligned} &[60P_3P_3'P_3'' - 1512P_3^4 - 70(P_3')^3]x_3^3 \\ &\qquad + [-180P_3^2P_3'' + 210P_3(P_3')^2]x_2x_3^2 = 0. \end{aligned}$$

The solution $x_3 = 0$ gives $x_2 = 0$, i. e. the point P_y. The other solution gives

$$x_2 = \omega(5\Theta_3'\Theta_8 - 756\Theta_3^4),\quad x_3 = \omega\cdot 15\Theta_3\Theta_8,$$

where ω is a proportionality factor, and where we have written

$$P_3 = \Theta_3,\quad 6P_3P_3'' - 7(P_3')^2 = \Theta_8,$$

since Θ_3 and Θ_8 reduce to these respective quantities under our assumption $P_2 = 0$. We have moreover assumed $P_3 \neq 0$, i. e. that

C_y is not a conic, in which case these considerations would be without value. Substituting into the expression for x_1, we find

$$x_1 = \omega \frac{7(5\Theta_3'\Theta_8 - 756\Theta_3^4)^2 + 25\Theta_8^3}{210\Theta_3\Theta_8}.$$

If then we put $\omega = 210\Theta_3\Theta_8$, we find, for the Halphen point of P_y, the expression

$$\begin{aligned}\sigma = &[7(5\Theta_3'\Theta_8 - 756\Theta_3^4)^2 + 25\Theta_8^3]y \\ &+ 210\Theta_3\Theta_8(5\Theta_3'\Theta_8 - 756\Theta_3^4)z + 15\cdot 210\Theta_3^2\Theta_8^2\varrho,\end{aligned}$$

under the assumption $P_2 = 0$.

But there must be a covariant, which for $P_2 = 0$ reduces to σ. We find that σ itself is not a covariant. The most general transformation of the independent variable converts σ into

$$\bar{\sigma} = \frac{1}{(\xi')^{24}}(\sigma + 1050\Theta_3^2\Theta_8^2\mu y).$$

But we have also

$$\bar{\Theta}_3^2\bar{\Theta}_8^2\bar{P}_2 = \frac{1}{(\xi')^{24}}\left(\Theta_3^2\Theta_8^2P_2 - \frac{2}{3}\Theta_3^2\Theta_8^2\mu\right),$$

so that

$$h = \sigma + 1575\Theta_3^2\Theta_8^2P_2 y$$

is a covariant. We have found therefore the following *covariant expression for the Halphen point which belongs to P_y*:

$$\begin{aligned}(29)\qquad h = &[7(5\Theta_3'\Theta_8 - 756\Theta_3^4)^2 + 25\Theta_8^3 + 1575\Theta_3^2\Theta_8^2P_2]y \\ &+ 210\Theta_3\Theta_8(5\Theta_3'\Theta_8 - 756\Theta_3^4)z + 3150\Theta_3^2\Theta_8^2\varrho.\end{aligned}$$

This expression shows that the *Halphen point* coincides with P_y, if and only if $\Theta_8 = 0$ *Halphen* has called such points of a curve which coincide with their Halphen point *coincidence points*. We shall investigate, in the next paragraph, those curves all of whose points are coincidence points

Here we will notice only that, according to (27) and (28), the osculating cubic in a coincidence point becomes

$$\Omega_1(x) = 0,$$

and that this cubic has a double point at P_y. We have therefore the remarkable result due to *Halphen:*

In a point of coincidence, the osculating cubic has a double point. In such a point there does not exist, as in other points of the curve, a cubic one of whose branches has a contact of the eighth order with it

In the general case, if $\Theta_8 \neq 0$, the cubic $\Omega_1(x) = 0$ is also of special interest. It is the only one of the eight-pointic cubics which has a double point at P_y. We shall call it the *eight-pointic nodal*

cubic. The two tangents of $\Omega_1 = 0$ at its double point, are the line $P_y P_z$, i. e. $x_3 = 0$, and

$$(30) \qquad 3P_3 x_2 - 2P_3' x_3 = 0.$$

They are always distinct if $P_3 \neq 0$, i. e. the eight-pointic nodal cubic can have a cusp only at such points of the curve C_y whose osculating conic hyperosculates the curve, in which case the cubic degenerates. If

$$B = 5\Theta_3' \Theta_8 - 756\Theta_3^4 = 0,$$

$P_y P_\varrho$ passes through the Halphen point. But we have, in general,

$$\bar{B} = \frac{1}{(\xi')^{12}}[B - 15\eta\Theta_3\Theta_8]$$

Therefore, if we make a transformation $\xi = \xi(x)$, for which

$$\eta = \frac{B}{15\Theta_3\Theta_8},$$

$P_y P_\varrho$ will pass through the Halphen point. The triangle $P_y P_z P_\varrho$ is determined uniquely by this condition. *We have therefore a complete system of geometrically interpreted covariants, in y, $\bar{z}$ and $\bar{\varrho}$, provided that* $\Theta_8 \neq 0$, if in the general expressions for $\bar{z}$ and $\bar{\varrho}$ the above value of η be substituted.

The tangent $P_y P_z$ intersects the osculating cubic again in a point, which is easily found to be

$$(31) \qquad \gamma = \left(567\Theta_3^3 - \frac{5}{2}\Theta_8'\right)y - 20\Theta_8 z.$$

By its means we obtain again a set of geometrically interpreted covariants. The tangent to the cubic at P_γ intersects the cubic again in a point P_δ, and that at P_δ in a point P_ε. The latter must coincide with the Halphen point, according to the known theory of cubic curves.[1] The conditions that the loci of P_h, P_γ, P_δ shall be straight lines, conics, or special curves of any kind may serve to characterize special classes of curves C_y.

§ 5. The curves, all of whose points are coincidence points.

If all of the points of C_y are coincidence points, Θ_8 vanishes identically. We shall assume that our differential equation has been reduced to the Halphen canonical form, so that we have the conditions

$$p_1 = 0, \quad \Theta_3 = 1, \quad \Theta_8 = 0,$$

whence

$$P_2 = 0, \quad P_3 = 1.$$

---

1) P_γ is the so-called tangential of P_y. For these theorems cf. *Salmon's Higher plane curves*, Chapter V.

The differential equation becomes very simple, viz.:

$$y^{(3)} + y = 0.$$

Let ω be a third root of unity:

$$\omega = -\frac{1}{2} + \frac{1}{2} i\sqrt{3}, \quad \omega^2 = -\frac{1}{2} - \frac{1}{2} i\sqrt{3}.$$

Then we have the following fundamental system of solutions

$$y_1 = e^{-x}, \quad y_2 = e^{-\omega x}, \quad y_3 = e^{-\omega^2 x},$$

whence

$$\frac{y_2}{y_1} = e^{\frac{3}{2}x}\left(\cos\frac{1}{2}\sqrt{3}\,x - i\sin\frac{1}{2}\sqrt{3}\,x\right),$$

$$\frac{y_3}{y_1} = e^{\frac{3}{2}x}\left(\cos\frac{1}{2}\sqrt{3}\,x + i\sin\frac{1}{2}\sqrt{3}\,x\right).$$

Put

$$\xi = \frac{y_2 + y_3}{2y_1}, \quad \eta = \frac{y_3 - y_2}{2iy_1}, \quad x = \frac{2}{3}\sqrt{3}\,\varphi,$$

and we shall find

$$\xi = e^{\varphi\sqrt{3}}\cos\varphi, \quad \eta = e^{\varphi\sqrt{3}}\sin\varphi, \quad \xi^2 + \eta^2 = e^{2\varphi\sqrt{3}} \tag{32}$$

Let ξ and η be cartesian coordinates. We notice that (32) represents a logarithmic spiral which intersects all of its radii at an angle of 30^0. Thence *Halphen*'s theorem: *any curve, all of whose points are coincidence points, may be obtained by projective transformation from a logarithmic spiral which intersects all of its radii at an angle of thirty degrees.*

Since we have

$$y' = z, \quad z' = \varrho, \quad \varrho' = -y,$$

and therefore also

$$z^{(3)} + z = 0, \quad \varrho^{(3)} + \varrho = 0,$$

we see that *each vertex of the triangle $P_y P_z P_\varrho$ describes a curve of coincidence points as well as P_y; moreover the locus of each of the vertices of the triangle is at the same time the envelope of one of the sides which ends there.*

In particular, if C_y coincides with (32), we find for C_z the cartesian equations

$$\xi_1 = e^{\varphi\sqrt{3}}\cos\left(\varphi + \frac{4\pi}{3}\right), \quad \eta_1 = e^{\varphi\sqrt{3}}\sin\left(\varphi + \frac{4\pi}{3}\right),$$

and for C_ϱ,

$$\xi_2 = e^{\varphi\sqrt{3}}\cos\left(\varphi + \frac{2\pi}{3}\right), \quad \eta_2 = e^{\varphi\sqrt{3}}\sin\left(\varphi + \frac{2\pi}{3}\right),$$

i. e. *all of the vertices of the triangle $P_y P_z P_\varrho$, which is equilateral in this case, and whose center is the origin, describe congruent logarithmic spirals which are obtained from each other by rotation through an angle of 120^0.*

Since we have $P_8 = 1$, the osculating cubic, which is at the same time the eight-pointic nodal cubic, becomes

$$F = 5\,(x_2^2 - 2x_1x_3)\,x_2 - 4x_3^3 = 0,$$

where the triangle of reference is equilateral. Since this triangle, remaining always equilateral, changes its magnitude as P_y describes the logarithmic spiral, while the equation of the cubic does not change its from, *the cubic always remains similar to itself.* We can make this clearer by introducing rectangular coordinates, ξ and η. We shall have

$$\frac{x_2}{x_1} = \xi - i\eta, \quad \frac{x_3}{x_1} = \xi + i\eta,$$

so that the equation of the cubic becomes

$$\xi^3 - 3\xi\eta^2 - 10\,(\xi^2 + \eta^2) - 9i\,(3\xi^2\eta - \eta^3) = 0,$$

which shows that the eight-pointic nodal cubic is not a real curve. Its only real points are

$$\xi = 0,\ \eta = 0;\ \ \xi = 10,\ \eta = 0;\ \ \xi = -5,\ \eta = +\sqrt{75};\ \ \xi = -5,\ \eta = -\sqrt{75},$$

or in homogeneous coordinates, x_1, x_2, x_3,

$$(1, 0, 0);\quad (1, 10, 10);\quad (1, -5 + 5i\sqrt{3}, -5 - 5i\sqrt{3});$$
$$(1, -5 - 5i\sqrt{3}, -5 + 5i\sqrt{3}).$$

The rectangular coordinates of a point of the spiral corresponding to the angle $\varphi + \lambda$, are

$$\xi' = e^{\lambda\sqrt{3}}(\xi\cos\lambda - \eta\sin\lambda), \quad \eta' = e^{\lambda\sqrt{3}}(\xi\sin\lambda + \eta\cos\lambda),$$

and we obtain the rectangular equation of its eight-pointic nodal cubic by substituting

$$\frac{x_2}{x_1} = \xi' - i\eta', \quad \frac{x_3}{x_1} = \xi' + i\eta'$$

in $F = 0$. We have proved our statement and have moreover found the ratio of magnification. It is equal to $e^{\lambda\sqrt{3}}$.

The cubic $\Omega_2 = 0$ becomes in our case, after division by 21,

$$5x_1x_2^2 - 10x_1^2x_3 - 2x_2x_3^2 = 0.$$

It is an equi-anharmonic cubic, i. e. the double-ratio of the four tangents, which can be drawn to it from any one of its points, is equi-anharmonic. This follows at once if the invariants of this ternary cubic be computed.[1]) We find $S = 0$, which proves our assertion. This cubic contains the three vertices of the triangle $P_yP_zP_\varrho$, and we find that its tangents at P_y, P_z, P_ϱ are respectively P_yP_z, P_zP_ϱ

1) *Salmon, Higher plane curves,* 3d edition, p. 191 and p. 200

and $P_\varrho P_y$, i. e. the triangle is at the same time inscribed in, and circumscribed about the cubic.

We thus find Halphen's further theorem:

Given a logarithmic spiral of 30 degrees. If we construct an equilateral triangle which has the pole of the spiral as center, and any point of the spiral as one vertex, this triangle is at the same time inscribed in and circumscribed about an equi-anharmonic cubic which has a contact of the seventh order with the spiral at the point considered.

Of course there are two other cubics of this kind corresponding to the other two vertices of the triangle.

We must add however, that these cubics are imaginary, as well as the eight-point nodal cubic.

The relation of the eight-point cubics to each other and to the curve C_y in a coincidence point may possibly give rise to a misunderstanding. It looks as though each of the eight-point cubics would then be a nine-point cubic, since the nine points of intersection of any two of them coincide with P_y. This paradox is easily explained if we remember that the only cubic, having nine-points of intersection with C_y coincident at P_y, is the eight-point nodal cubic. Each of the other eight-point cubics intersects one of its branches in eight coincident points and the other branch in the remaining ninth point which also coincides with P_y. These cubics have with each other a contact of the eighth order, but only that one which has a double point at P_y has nine coincident points of intersection with C_y at P_y, and may therefore (improperly) be said to have contact of the eighth order with C_y.

§ 6. Curves of the third order.

If we assume $p_1 = p_2 = 0$, equations (20) shows that the osculating cubic hyperosculates the curve C_y at P_y, i. e. has more than nine consecutive points in common with it, if

$$(33)\quad \begin{aligned} 21&(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)(20P_3'P_3'' - 15P_3P_3^{(3)} + 63P_3^3) \\ &+ 100\{6P_3P_3'' - 7(P_3')^2\}\{7P_3P_3^{(4)} - 12(P_3'')^2 \\ &- 882P_3^2P_3'\} = 0. \end{aligned}$$

In order that C_y may itself be a cubic, it is clearly necessary and sufficient that this condition be fulfilled at all points, i. e. for all values of x.

The left member of (33) must be the special form assumed by a certain invariant Θ_{18} under the assumption $P_2 = 0$, $p_1 = 0$. We wish to find the general expression of this invariant.

Since $P_3^{(4)}$ is the highest derivative of P_3 which occurs in (33), the left member must be expressible in terms of $\Theta_3, \Theta_8, \Theta_{12}, \Theta_{16}, \Theta_{21}$. We find for $P_2 = 0$,

$$
\begin{aligned}
&\Theta_3 = P_3, \quad \Theta_8 = 6P_3P_3'' - 7(P_3')^2,\\
&\Theta_{12} = 18P_3^2P_3^{(3)} - 72P_3P_3'P_3'' + 56(P_3')^3,\\
(34)\qquad &\Theta_{16} = 18P_3^3P_3^{(4)} - 108P_3^2P_3'P_3^{(3)} - 72P_3^2(P_3'')^2\\
&\qquad\qquad + 384P_3(P_3')^2P_3'' - 224(P_3')^4,\\
&\frac{\Theta_{21}}{9\Theta_3} = 2\Theta_8[2P_3P_3^{(4)} + 4P_3'P_3^{(3)} - 8(P_3'')^2] - (\Theta_8')^2,
\end{aligned}
$$

whence

$$(35)\qquad \frac{7\Theta_{16}+32\Theta_8^2}{\Theta_3^2} = 126P_3P_3^{(4)} - 756P_3'P_3^{(3)} + 648(P_3'')^2.$$

On the other hand, we find

$$
\begin{aligned}
4\Theta_{18} = &-21\cdot 25(\Theta_8')^2 - 84\cdot 567\cdot 63\Theta_3^6 + 900\cdot 49\Theta_3^2\Theta_{12}\\
&+ 400\Theta_8[7P_3P_3^{(4)} - 12(P_3'')^2]
\end{aligned}
$$

or

$$
\begin{aligned}
t_{18} &= 4\Theta_{18} + 84\cdot 567\cdot 63\Theta_3^6 - 900\cdot 49\Theta_3^2\Theta_{12}\\
&= -21\cdot 25(\Theta_8')^2 + 400\Theta_8[7P_3P_3^{(4)} - 12(P_3'')^2],
\end{aligned}
$$

whence

$$t_{18} - \frac{175}{3}\frac{\Theta_{21}}{\Theta_3} - \frac{50}{9}\frac{7\Theta_{16}+32\Theta_8^2}{\Theta_3^2}\Theta_8 = 0,$$

or

$$
\begin{aligned}
(36)\quad &2^2\cdot 3^2\Theta_3^2\Theta_{18} + 2^2\cdot 3^9\cdot 7^3\Theta_3^8 - 2^2\cdot 3^4\cdot 5^2\cdot 7^2\Theta_3^4\Theta_{12} - 3\cdot 5^2\cdot 7\Theta_3\Theta_{21}\\
&\qquad - 2\cdot 5^2\Theta_8(7\Theta_{16} + 2^5\Theta_8^2) = 0,
\end{aligned}
$$

which, on account of the syzygy (12), may also be written

$$
\begin{aligned}
(37)\quad &2^2\cdot 3^2\Theta_3^2\Theta_{18} + 2^2\cdot 3^9\cdot 7^3\Theta_3^8 - 2^2\cdot 3^4\cdot 5^2\cdot 7^2\Theta_3^4\Theta_{12} + 3\cdot 5^2\cdot 7\Theta_{12}^2\\
&\qquad - 2^3\cdot 5^2\cdot 7\Theta_8\Theta_{16} - 2^6\cdot 5^2\Theta_8^3 = 0.
\end{aligned}
$$

The condition for a cubic is $\Theta_{18} = 0$, *where* Θ_{18} *is given by* (36) *or* (37).

If the absolute invariant $\frac{\Theta_8^3}{\Theta_3^8}$ is a constant, the differential equation may be transformed into one with constant coefficients. The curve is then what Halphen calls an *anharmonic curve* (cf. § 8). It may be, at the same time, a cubic. In fact in this case Θ_{12} must vanish, and hence also Θ_{16} and Θ_{21}. The condition $\Theta_{18} = 0$ reduces therefore to

$$(38)\qquad 2^4\cdot 5^2\Theta_8^3 - 3^9\cdot 7^3\Theta_3^8 = 0.$$

The cubic is then a cuspidal cubic; as may be seen by setting up the corresponding differential equation. We may also verify this directly as follows. Put the equations of the cubic into the form

$$y_1 = x^{\frac{3}{2}}, \quad y_2 = x, \quad y_3 = 1.$$

The differential equation, of which these functions form a fundamental system, is found to be

$$y^{(3)} + \frac{1}{2x} y'' = 0.$$

On computing the invariants we find

$$\Theta_3 = \frac{5}{2\cdot 3^3}\,\frac{1}{x^3}, \quad \Theta_8 = \frac{5^2\cdot 7}{2^4\cdot 3^5}\,\frac{1}{x^8},$$

whence we again conclude (38). Therefore (38) is the condition for a cuspidal cubic.

It will be advisable to set up also the condition for a nodal cubic. Take the cubic in the form

$$y_1 = 1, \quad y_2 = x, \quad y_3 = -x^2 + (1-x)^{-1},$$

whence we deduce the differential equation

$$y^{(3)} - \frac{3}{(1-x)\,[1-(1-x)^3]}\, y'' = 0,$$

for which

$$p_1 = \frac{-1}{(1-x)\,[1-(1-x)^3]}, \quad p_2 = 0, \quad p_3 = 0.$$

It is convenient to write $t = (1-x)^{-1}$, so that

$$p_1 = \frac{t^4}{1-t^3}.$$

On computing the invariants we find

$$\Theta_3 = 10\,\frac{t^9+t^6}{(1-t^3)^3}, \quad \Theta_8 = \frac{60^2\cdot 7\,t^{17}(1+t^3+t^6)}{(1-t^3)^8}, \quad \Theta_{12} = \frac{60^2\cdot 7\cdot 30\,t^{24}}{(1-t^3)^8},$$

whence

$$\lambda = \frac{\Theta_8^{\,3}}{\Theta_3^{\,8}} = \frac{6^6\cdot 7^3}{10^2}\,\frac{t^3(1+t^3+t^6)^3}{(1+t^3)^8}, \quad \mu = \frac{\Theta_{12}}{\Theta_3^{\,4}} = \frac{6^2\cdot 7\cdot 3}{10}\left(\frac{1-t^3}{1+t^3}\right)^4.$$

If we put

(39) $$\lambda = \frac{2^4\cdot 3^6\cdot 7^3}{5^2}\,\eta, \quad \mu = -\frac{2\cdot 3^2\cdot 7}{5}\,\xi,$$

so as to have the same notation as *Halphen*, we find

$$\eta = \frac{z(1+z+z^2)^3}{(1+z)^8}, \quad \xi = -3\left(\frac{1-z}{1+z}\right)^4,$$

where $z = t^3$. If we eliminate z *we find*

(40) $$(2^8\cdot 3^2\eta + \xi^2 - 2\cdot 3^3\xi - 3^5)^2 + 2^6\cdot 3\cdot \xi^3 = 0,$$

as the condition for a nodal cubic.

In general a ternary cubic has an absolute invariant. We wish to find the condition for a cubic curve whose invariant has a given value. Clearly it must be possible to obtain this condition from $\Theta_{18} = 0$ by integration.

Let us, then, assume $\Theta_{18} = 0$, and introduce absolute invariants by putting

$$\lambda = \frac{\Theta_8{}^3}{\Theta_3{}^8}, \quad \mu = \frac{\Theta_{12}}{\Theta_3{}^4},$$

whence

$$\lambda' = \frac{\Theta_8{}^2\Theta_{12}}{\Theta_3{}^9}, \quad \mu' = \frac{\Theta_{16}}{\Theta_3{}^5}, \quad \frac{d}{dx}\left(\frac{\lambda}{\mu^2}\right) = -\frac{\Theta_8{}^2\Theta_{21}}{\Theta_{12}{}^3}.$$

If we divide (37) by $\Theta_3{}^8$ and introduce these quantities, the condition $\Theta_{18} = 0$ becomes

$$[21\{5^2\mu^2 - 2^2\cdot 3^2\cdot 5^2\cdot 7\mu + 2^2\cdot 3^8\cdot 7^2\} - 2^6\cdot 5^2\lambda]\,\lambda' - 2^3\cdot 5^2\cdot 7\,\lambda\mu\mu' = 0.$$

So as to have the same notation as Halphen, put again

$$\lambda = \frac{2^4\cdot 3^6\cdot 7^3}{5^2}\eta, \quad \mu = -\frac{2\cdot 3^2\cdot 7}{5}\xi,$$

so that

(41) $$\eta = \frac{5^2}{2^4\cdot 3^6\cdot 7^3}\frac{\Theta_8{}^3}{\Theta_3{}^8}, \quad \xi = -\frac{5}{2\cdot 3^2\cdot 7}\frac{\Theta_{12}}{\Theta_3{}^4}.$$

Then, the above equation becomes

(42) $$\xi\eta\xi' = \frac{3}{8}\left[(\xi+3)(\xi+27) - 2^4\cdot 3\eta\right]\eta'.$$

Put further

$$\frac{3}{8}(\xi+3)(\xi+27) = A, \quad 3^2\cdot 2^5\eta = \zeta,$$

so that (42) becomes

(43) $$\zeta\xi\xi' + (\zeta - A)\zeta' = 0.$$

Halphen has shown that the general integral of this equation is

(44) $$P\zeta^3 = cQ^3,$$

where c is an arbitrary constant, and where

(45) $$\begin{aligned} 2^6P &= (2^3\zeta + \xi^2 - 2\cdot 3^3\xi - 3^5)^2 + 2^6\cdot 3\xi^3, \\ 2^6Q &= 2^6\zeta^2 + 2^4(\xi+3^3)(\xi - 3^2\cdot 5)\zeta + (\xi+3^3)^4. \end{aligned}$$

The equations $P=0$, $Q=0$ give special solutions of (41). The case $P=0$ corresponds to the case of a nodal cubic, as we may verify at once.

We proceed to explain Halphen's integration of equation (43) The geometrical significance of this equation makes it certain that its general integral is algebraic. Let

$$P = \zeta^n + M_1\zeta^{n-1} + M_2\zeta^{n-2} + \cdots + M_{n-1}\zeta + M_n = 0,$$

where M_k is a rational function of ξ, be an integral of this equation. Interpret, for a moment, ξ and ζ as cartesian coordinates of a point in a plane. Equation (43) determines the ratio of ξ' to ζ' for every value of ξ and ζ, i. e. it determines the direction of the tangent of any integral curve at any point of the plane. If $P=0$ is an integral curve, the direction of its tangent at any one of its points must therefore coincide with that found from (43). But the direction of

the tangent of this curve, at one of its points, will be found from the equations

$$P = 0, \quad \frac{\partial P}{\partial \xi}\xi' + \frac{\partial P}{\partial \zeta}\zeta' = 0.$$

The expression

$$\frac{\partial P}{\partial \xi}\xi' + \frac{\partial P}{\partial \zeta}\zeta',$$

which, on account of (43), is proportional to

$$\xi\zeta\frac{\partial P}{\partial \zeta} + (A - \zeta)\frac{\partial P}{\partial \xi},$$

must therefore vanish as a consequence of $P = 0$. Since this expression is again an integral rational function of ζ of the n^{th} degree, with coefficients rational in ξ, we must have

$$\xi\zeta\frac{\partial P}{\partial \zeta} + (A - \zeta)\frac{\partial P}{\partial \xi} + \lambda P = 0,$$

where λ is a function of ξ. In this equation ξ and ζ are regarded as independent variables. Denote by $M_1', M_2', \ldots$ the derivatives of $M_1, M_2, \ldots$ with respect to ξ. We find

$$\begin{aligned}
M_1' &= \lambda + n\xi, \\
M_2' &= [\lambda + (n-1)\xi]\, M_1 + AM_1', \\
M_3' &= [\lambda + (n-2)\xi]\, M_2 + AM_2', \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
M_n' &= (\lambda + \xi)\, M_{n-1} + AM_{n-1}', \\
0 &= \lambda M_n + AM_n'.
\end{aligned}$$

Suppose that M_n is a fraction $\frac{\alpha}{\beta}$, where α and β are integral rational functions of ξ. The last equation will then give

$$\lambda = A\frac{\beta'}{\beta} - A\frac{\alpha'}{\alpha},$$

i. e. the factors of the denominator of λ will be simple. But, in that case, the first equation of our system would make M_1 a transcendental function of ξ, contrary to our hypothesis. Therefore, M_n and λ must be integral rational functions of ξ. Moreover, the expression

$$\lambda = -A\frac{M_n'}{M_n}$$

shows, that λ must be of the first degree, since A is quadratic. We see at once that M_1 must be of the second, M_2 of the 4^{th}, ... M_n of the $2n^{\text{th}}$ degree.

We know one solution of (43) corresponding to $n = 2$. It is the equation (40) which gives the condition for a nodal cubic. If we introduce ζ instead of η, this solution may be written

$$2^6 P = (2^3\zeta + \xi^2 - 2\cdot 3^3\xi - 3^5)^2 + 2^6\cdot 3\,\xi^3 = 0.$$

The corresponding function λ is found to be

$$\lambda = -\frac{3}{2}(\xi + 9).$$

We may find another particular integral, also corresponding to $n = 2$, by trial. It is

$$2^6 Q = 2^6 \zeta^2 + 2^4 (\xi + 3^3)(\xi - 3^2 \cdot 5)\zeta + (\xi + 3^3)^4,$$

for which

$$\lambda = -\frac{3}{2}(\xi + 3).$$

We have, therefore

$$\xi\zeta\frac{\partial P}{\partial \zeta} + (A - \zeta)\frac{\partial P}{\partial \xi} = \frac{3}{2}(\xi + 9)P,$$

$$\xi\zeta\frac{\partial Q}{\partial \zeta} + (A - \zeta)\frac{\partial Q}{\partial \xi} = \frac{3}{2}(\xi + 3)Q,$$

which equations may be written

$$\xi\zeta\frac{\partial (P\zeta^3)}{\partial \zeta} + (A - \zeta)\frac{\partial (P\zeta^3)}{\partial \xi} = \frac{9}{2}(\xi + 3)P\zeta^3,$$

$$\xi\zeta\frac{\partial (Q^3)}{\partial \zeta} + (A - \zeta)\frac{\partial (Q^3)}{\partial \xi} = \frac{9}{2}(\xi + 3)Q^3.$$

But precisely the same equation will be satisfied by

$$P\zeta^3 - cQ^3,$$

where c is an arbitrary constant. Therefore, the general integral of (43) is

$$P\zeta^3 = cQ^3,$$

which is the result we wished to prove.

We have noticed already that $P = 0$ corresponds to the case of a nodal cubic. The significance of $Q = 0$ will appear very soon. Another special solution is of importance. For a certain value of c, $P\zeta^3 - cQ^3$ will be a perfect square In fact

$$Q^3 + 2^3 \cdot 3^{[illegible]} \cdot \zeta^3 P = R^2,$$

where

(46) $$2^9 R = 2^9 \zeta^3 + 2^6 \cdot 3[(\xi - 3^2)^2 + 2^4 \cdot 3^4]\zeta^2 + 2^3 \cdot 3(\xi + 3^3)^3(\xi - 3^2 \cdot 5)\zeta + (\xi + 3^3)^6.$$

The general solution of (43) may therefore be written

$$R^2 + hQ^3 = 0,$$

where h is the arbitrary constant.

We may on the other hand consider the algebraic equation of the cubic, which we have deduced explicitly under the assumption of the special choice of coordinates involved in the reduction of the differential equation to the *Laguerre-Forsyth* form. A ternary cubic

has two relative invariants S and T of degrees 4 and 6 respectively in the coefficients of the cubic. The absolute invariant is

$$\frac{T^2}{S^3} = k,$$

and the discriminant is $T^2 + 64S^3$.[1] If we observe the explicit equation of the cubic we notice that the highest derivative of P_8 which occurs in it is $P_8^{(8)}$, which can enter into S and T only by means of the invariant Θ_{12}. Therefore, the highest powers of Θ_{12} which can occur in S or T are respectively the fourth and the sixth. But, of the two quantities ξ and η, only ξ contains Θ_{12}, and moreover, linearly. Therefore in terms of ξ and η S must contain no higher power of ξ than the fourth, and T no higher than the sixth. But Q and R contain precisely these powers of ξ, so that except for numerical factors Q and R must be proportional to S and T respectively. Finally, since we obtain a nodal cubic if

$$T^2 + 64S^3 = 0,$$

and also if $P = 0$, the case $k = -64$ must correspond to $h = -1$. Since h and k can differ only by a numerical factor we find

$$k = 64h.$$

If we apply the known results of the theory of invariants of a ternary cubic, we find therefore the following result.

Let T and S be the fundamental invariants of a ternary cubic. Put

$$\frac{T^2}{S^3} = 64h.$$

Then the condition, that the differential equation (1) *shall represent a cubic curve with the absolute invariant h, is*

$$R^2 + hQ^3 = 0, \tag{47}$$

which may also be written

$$2^3 \cdot 3^5 \cdot \xi^3 P + (h+1) Q^3 = 0$$

In particular $P = 0$, $Q = 0$, $R = 0$ are the conditions for a nodal, an equi-anharmonic, an harmonic cubic respectively.

It will be noticed that an extensive theory may be developed for the general curve C_y, based upon the theory of the covariants of the osculating cubic Moreover, since we have shown how to compute the invariants S and T, there remain no serious difficulties to overcome. As x changes, each of the points of inflection of the osculating cubic describes a curve, its Hessian and other covariants envelope curves, etc.... Special properties of these various covariant curves will serve to characterize special classes of plane curves C_y. There

1) *Salmon's Higher Plane Curves*, Chapter V, Section V.

are further the simultaneous covariants of osculating conic and cubic, all of which promise interesting results. We may also regard the determination of the osculating cubic as an approximate integration of the differential equation.

We have seen in § 1 that the invariants of the *Lagrange adjoint* equation differ from those of (1) only by having $-\Theta_3$ in place of Θ_3, and therefore $-\Theta_{12}$ and $-\Theta_{21}$ in place of Θ_{12} and Θ_{21}. But the integral curves of a linear differential equation and its Lagrange adjoint are dualistic to each other. It is a very easy matter, therefore, to find from the equations of this paragraph, the conditions for curves of the third class. We may also by considering the adjoint equation, determine the osculating curves of the third class for an arbitrary curve, etc....

The curves of the third order without a double point, being of deficiency unity, may be studied by means of elliptic functions.[1] We shall follow *Halphen* in giving a brief treatment of the curves from this point of view, culminating in the determination of the number of coincidence points upon a cubic curve. This theory rests upon the following theorem:

Let the coordinates x and y of a point of a plane curve be given as uniform doubly periodic functions of an argument t, the periods of these two functions and their poles being identical. The locus of the point (x, y) will be an algebraic curve, whose degree is equal to the number of poles of the doubly-periodic functions in a period-parallelogram.

To prove this theorem we make use of a method of representation for the elliptic functions due to *Hermite*. Put

$$q = e^{-\pi \frac{k}{K}}, \quad H(t) = 2\sum_{m=0}^{\infty}(-1)^m q^{\left(\frac{2m+1}{2}\right)^2} \sin\frac{(2m+1)\pi t}{2K},$$

$$Z(t) = \frac{H'(t)}{H(t)},$$

making use of the ordinary notation for the Jacobian functions.

Then we shall have

$$x = u + a\,Z(t-\alpha) + b\,Z(t-\beta) + c\,Z(t-\gamma) + \cdots$$
$$y = u' + a'Z(t-\alpha) + b'Z(t-\beta) + c'Z(t-\gamma) + \cdots$$

where $\alpha, \beta, \gamma, \ldots$ are the poles of the two elliptic functions. The constants $a, b, \ldots a', b', \ldots$, which are the residuals of these functions at their poles, satisfy the conditions

---

1) *Clebsch* first indicated the importance of elliptic functions for the theory of curves of deficiency unity.

$$a + b + c + \cdots = a' + b' + c' + \cdots = 0.$$

The periods of the elliptic functions will be $2K$ and $2iK'$.

In order to find the intersections of the curve, represented in this way, with an arbitrary straight line

$$Ax + By + C = 0,$$

we need only find the values of the argument t for which the elliptic function $Ax + By + C$ vanishes. But this function has the same periods $2K$ and $2iK'$, and the same poles $\alpha, \beta, \gamma, \ldots$ as x and y. In a period-parallelogram a doubly periodic function has a many zeros as poles; the values of the argument which differ from each other by mere multiples of the periods correspond to the same point of the curve. Hence the function $Ax + By + C$ vanishes for as many non-congruent values of t as x and y have poles in a period-parallelogram, whence follows the theorem which we were to prove.

Make an arbitrary projective transformation

$$X = \frac{A_1 x + A_1' y + B_1}{Ax + A'y + B}, \quad Y = \frac{A_2 x + A_2' y + B_2}{Ax + A'y + B}.$$

X and Y will be two doubly periodic functions of t with the periods $2K$ and $2iK'$. The poles of these functions will be different from $\alpha, \beta, \gamma, \ldots$ but equal to them in number. X and Y may, therefore, be represented by elliptic functions in the same way as before, making use of the same Z function, only the constants $a, a, b, \ldots \alpha, \beta, \ldots$ being changed. Therefore, *this representation by elliptic functions has a projective character. Moreover the quantity q, or the ratio of the periods, is an absolute invariant.*

Let there be only three poles α, β, γ, so that the curve is a cubic. Of the thirteen constants, which appear to enter the equations, only nine are independent. On the other hand, a plane cubic is determined by nine conditions. We shall prove that any non-singular cubic may be represented in this way In order to do this, it becomes necessary to study the curve somewhat more in detail

According to a well-known theorem, due to *Liouville*, the sum of the zeros of a doubly periodic function in a period-parallelogram can differ from the sum of its poles only by multiples of the periods. We find at once the following theorem:

The three values of t, which correspond to the intersections of the cubic with a straight line, have a sum which can differ from $\alpha + \beta + \gamma$ only by multiples of the periods.

This gives rise to the following corollaries:

The nine values of the argument t

$$\frac{\alpha + \beta + \gamma}{3} + \frac{2pK + 2p'iK'}{3},$$

where p and p' are integers, correspond to the points of inflection.

Through an arbitrary point of the curve, four straight lines may be drawn which shall be tangent to it in other points. If t is the argument corresponding to the first point, the arguments of the four points of contact are

$$\frac{\alpha+\beta+\gamma-t}{2}+pK+p'iK'.$$

But these properties of the curve prove it to be of the sixth class, i. e. a non-singular cubic. To prove, on the other hand, that every cubic of the sixth class may be represented in this way, we consider a special case. Write

$$x=Z(t)-Z(t+K),$$
$$y=\frac{i\pi}{2K}+Z(t+iK')-Z(t+K).$$

We proceed to eliminate t. We have

$$Z(t)=\frac{H'(t)}{H(t)},\quad Z(t+K)=\frac{H_1'(t)}{H_1(t)},\quad Z(t+iK')=-\frac{i\pi}{2K}+\frac{\Theta'(t)}{\Theta(t)},$$

making use of the ordinary notation for the Jacobian Θ functions. We find, therefore

$$x=\frac{d}{dt}(\log snt-\log cnt)=\frac{dnt}{snt\,cnt},$$
$$y=-\frac{d}{dt}(\log cnt)=\frac{snt\,dnt}{cnt},$$

whence

$$xy(x-y)=x-k^2y,$$

where k is the modulus of the elliptic functions, which is connected with q by the equation

$$\sqrt{k}=\frac{2\sqrt[4]{q}+2\sqrt[4]{q^9}+\cdots}{1+2q+2q^4+\cdots}.$$

The equation, made homogeneous, becomes

$$xy(x-y)=z^2(x-k^2y).$$

But any non-singular cubic may be reduced to this form if the vertex $x=0$, $y=0$ of the triangle of reference be taken as a point of inflection, while the other two coincide with the points of contact of two of the tangents which may be drawn through this point. Such a triangle exists, if and only if the cubic is of the sixth class. We have shown, therefore, that *every cubic of the sixth class may be represented by elliptic functions in the way indicated.*

Consider, now, a function of the third degree in x and y. It is a doubly periodic function of t, whose poles α, β, γ are triple. We see, therefore, that *the arguments which correspond to the intersections of the cubic with another cubic differ from $3(\alpha+\beta+\gamma)$ only by multiples of the periods.*

Further: *if we construct in a point, whose argument is t, the pencil of all cubics having with the given cubic contact of the seventh order in the given point, these cubics have a further point in common, whose argument is*

$$3(\alpha+\beta+\gamma)-8t.$$

At every point of the cubic, whose argument is of the form

$$\frac{\alpha+\beta+\gamma}{3}+\frac{2pK+2p'iK'}{9},$$

there exists a pencil of cubics having contact of the eighth order with the given cubic at this point. These points are the coincidence points of the cubic.

Of these 81 points, however, only 72 are truly coincidence points. For the 9 points of inflection are included among them. The existence of 72 coincidence points on a non-singular cubic has also been demonstrated by *Halphen* in another way.[1])

The results found so far, enable us to verify the further theorem: *On a cubic the coincidence points may be grouped in triangles which are at the same time inscribed in, and circumscribed about the curve.* For, if we compute the argument of the point where a tangent at a point of coincidence again intersects the cubic, we find that this point is again a coincidence point.

§ 7. Canonical development for the equation of a plane curve in non-homogeneous coordinates.

If $P_2=0$, the equation of the osculating conic is

$$x_2^2-2x_1x_3=0,$$

or if we introduce non-homogeneous coordinates,

$$\eta=\frac{1}{2}\xi^2,$$

where

$$\xi=\frac{x_2}{x_1},\quad \eta=\frac{x_3}{x_1}.$$

Since C_y has contact of the 4th order with this conic, we must have similarly for C_y a development of the form

$$\eta=\frac{1}{2}\xi^2+a_5\xi^5+a_6\xi^6+a_7\xi^7+\cdots,$$

where

$$\xi=\frac{y_2}{y_1},\quad \eta=\frac{y_3}{y_1}.$$

1) *Halphen*, Journal de Mathématiques, 3e série, t. II, p 376.

We can easily find this development. We have from (16) and (17)

$$\xi^2 - 2\eta = \frac{y_2^2 - 2y_1 y_3}{y_1^2} = \frac{1}{10} P_3 x^5 + \frac{1}{60} P_3' x^6 + \frac{1}{420} P_3'' x^7 + \cdots,$$

if we write only the terms up to and including the seventh order. We have further

$$\xi = \frac{y_2}{y_1} = x\left(1 + \frac{1}{8} P_3 x^3 + \cdots\right),$$

so that up to the order of terms here retained,

$$\xi^5 = x^5, \quad \xi^6 = x^6, \quad \xi^7 = x^7.$$

We find therefore

$$\eta = \frac{1}{2}\xi^2 - \frac{1}{20} P_3 \xi^5 - \frac{1}{120} P'_3 \xi^6 - \frac{1}{840} P''_3 \xi^7 + \cdots \tag{48}$$

But we can simplify this development considerably. The form of the series

$$\eta = \frac{1}{2}\xi^2 + a_5 \xi^5 + a_6 \xi^6 + a_7 \xi^7 + \cdots$$

will not be changed if we make the most general projective transformation which converts the osculating conic, as well as the point P_y and the tangent $P_y P_z$ into themselves. This merely amounts to taking as triangle of reference the most general triangle of which P_y shall be a vertex and $P_y P_z$ a side and which shall be self-conjugate with respect to the osculating conic. This projective transformation is

$$X = \frac{\beta\xi + \sqrt{2}\alpha\beta\eta}{1 + \sqrt{2}\alpha\xi + \alpha^2\eta} = \frac{\lambda}{\nu}, \quad Y = \frac{\beta^2\eta}{1 + \sqrt{2}\alpha\xi + \alpha^2\eta} = \frac{\mu}{\nu},$$

where

$$\lambda = \beta\xi + \sqrt{2}\alpha\beta\eta, \quad \mu = \beta^2\eta, \quad \nu = 1 + \sqrt{2}\alpha\xi + \alpha^2\eta.$$

If we substitute the development (48) for η, we find

$$\begin{aligned}
\lambda &= \varrho + \sqrt{2}\alpha\beta\,(a_5\xi^5 + a_6\xi^6 + a_7\xi^7 + \cdots),\\
\mu &= \sigma + \beta^2\,(a_5\xi^5 + a_6\xi^6 + a_7\xi^7 + \cdots),\\
\nu &= \tau + \alpha^2\,(a_5\xi^5 + a_6\xi^6 + a_7\xi^7 + \cdots),
\end{aligned}$$

where

$$\begin{aligned}
\varrho &= \beta\xi + \frac{1}{2}\sqrt{2}\alpha\beta\xi^2,\\
\sigma &= \frac{1}{2}\beta^2\xi^2,\\
\tau &= 1 + \sqrt{2}\alpha\xi + \frac{1}{2}\alpha^2\xi^2,
\end{aligned}$$

and, of course,

$$\varrho^2 - 2\sigma\tau = 0.$$

We find therefore

$$\lambda^2 - 2\mu\nu = 2\left(\sqrt{2}\alpha\beta\varrho - \beta^2\tau - \alpha^2\sigma\right)\left(a_5\xi^5 + a_6\xi^6 + a_7\xi^7 + \cdots\right),$$

whence

$$\frac{\lambda^2 - 2\mu\nu}{\nu^2} = -2\beta^2\left[a_5\xi^5 + (a_6 - 2\sqrt{2}a_5\alpha)\,\xi^6 \right. \tag{49}$$
$$\left. + (a_7 - 2\sqrt{2}\alpha a_6 + 5\alpha^2 a^5)\,\xi^7 + \cdots\right].$$

Further we have

$$X = \frac{\beta\xi\left[1 + \frac{1}{2}\sqrt{2}\alpha\xi + (\xi^4)\right]}{1 + \sqrt{2}\alpha\xi + \frac{1}{2}\alpha^2\xi^2 + (\xi^5)},$$

where (ξ^k) denotes an aggregate of terms of at least the k'th order.

Consequently

$$X = \beta\xi\left[1 - \frac{1}{2}\sqrt{2}\alpha\xi + \frac{1}{2}\alpha^2\xi^2 + \cdots\right],$$

so that we shall have, exact to terms of the seventh order,

$$X^7 = \beta^7\xi^7,$$

$$X^6 = \beta^6\,(\xi^6 - 3\sqrt{2}\alpha\xi^7),$$

$$X^5 = \beta^5\left(\xi^5 - \frac{5}{2}\sqrt{2}\alpha\xi^6 + \frac{15}{2}\alpha^2\xi^7\right),$$

whence

$$\xi^7 = \frac{1}{\beta^7}X^7,$$

$$\xi^6 = \frac{1}{\beta^6}\left(X^6 + 3\sqrt{2}\frac{\alpha}{\beta}X^7\right),$$

$$\xi^5 = \frac{1}{\beta^5}\left(X^5 + \frac{5}{2}\sqrt{2}\frac{\alpha}{\beta}X^6 + \frac{15}{2}\frac{\alpha^2}{\beta^2}X^7\right).$$

Substituting in (49) we find

$$Y = \frac{1}{2}X^2 + \frac{a_5}{\beta^3}X^5 + \frac{1}{\beta^4}\left(\frac{1}{2}\sqrt{2}a_5\alpha + a_6\right)X^6$$
$$+ \frac{1}{\beta^6}\left(a_7 + \sqrt{2}a_6\alpha + \frac{1}{2}a_5\alpha^2\right)X^7 + \cdots.$$

But we have the two constants α and β at our disposal. Let us choose them so that

$$\frac{a_5}{\beta^3} = 1, \quad a_6 + \frac{1}{2}\sqrt{2}a_5\alpha = 0.$$

We thus find the canonical form

$$Y = \frac{1}{2}X^2 + X^5 + A_7X^7 + \cdots \tag{50}$$

for the development. Since we have

$$a_5 = -\frac{1}{20} P_3, \quad a_6 = -\frac{1}{120} P_3', \quad a_7 = -\frac{1}{840} P''_3,$$

the value of A_7 will be:

(51) $$A_7 = \frac{(-20)^{\frac{8}{3}}}{100800} \frac{\Theta_8}{\Theta_3^{\frac{8}{3}}}.$$

All of the coefficients will be absolute invariants.

It is clear that such a development always exists, except for such points of the curve C_y whose osculating conic hyperosculates it. It only remains to determine the geometrical significance of this canonical form. Since we have

$$\alpha = -\frac{\sqrt{2}\,a_6}{a_5} = -\frac{1}{6}\sqrt{2}\,\frac{P_3'}{P_3}, \quad \beta = a_5^{\frac{1}{3}} = \left(-\frac{P_3}{20}\right)^{\frac{1}{3}},$$

the line $X = 0$ is, in our original system of coordinates

$$3 P_3 x_2 - P_3' x_3 = 0,$$

which is nothing more or less than the polar of the point $(P_3', 3P_3, 0)$ with respect to the osculating conic. But the covariant expression for this point is

$$C_4' = \Theta_3' y + 3\Theta_3 z.$$

Therefore, *if the differential equation be reduced to the Halphen canonical form* ($\Theta_3 = 1$), *the corresponding point P_z will be the second vertex of that triangle of reference for which the development becomes canonical. In general, the second vertex of the canonical triangle is given by the covariant C_4'.*

The transformation of coordinates, which we have just made, may be written in homogeneous form

$$\begin{aligned}
\overline{x}_1 &= x_1 + \sqrt{2}\,\alpha x_2 + \alpha^2 x_3, \\
\overline{x}_2 &= \qquad \beta x_2 + \sqrt{2}\,\alpha\beta x_3, \\
\overline{x}_3 &= \qquad\qquad \beta^2 x_3.
\end{aligned}$$

If we apply this transformation to the equation $\Omega_1(x) = 0$ of the eight-pointic nodal cubic, we find after dividing by a numerical factor

$$F = \overline{x}_2^3 + 16\overline{x}_3^3 - 2\overline{x}_1\overline{x}_2\overline{x}_3 = 0.$$

The *Hessian* of F, again omitting a numerical factor, is

$$H = -3\overline{x}_2^3 - 48\overline{x}_3^3 - 2\overline{x}_1\overline{x}_2\overline{x}_3 = 0.$$

For the three points of inflection of $F = 0$, we find therefore

$$\overline{x}_1 = 0, \quad \overline{x}_2 : \overline{x}_3 = -\sqrt[3]{16} \text{ or } -\omega\sqrt[3]{16} \text{ or } -\omega^2\sqrt[3]{16},$$[1])

where ω is a cube root of unity.

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1) The Hessian of a plane curve intersects it in its double points and points of inflection. cf. *Salmon's Higher Plane curves.*

We have the following result.

In order to obtain the canonical form (50) *of the development, the triangle of reference must be chosen as follows. One vertex is a point on the curve and one side of the triangle is the tangent at this point. The second side is the line upon which are situated the three points of inflection of the eight-pointic nodal cubic. The third side is the polar of the intersection of the other two with respect to the osculating conic. The numerical factors, which still remain arbitrary in a projective system of coordinates after the triangle of reference has been chosen, must be determined in such a way that the coordinates of one of the three points of inflection of the eight-pointic nodal cubic shall be* $(0, -\sqrt[3]{16}, 1)$, *and that the coordinates of the tangent to the cubic at this point shall be* $(2\sqrt[3]{16},\ 3\sqrt[3]{16^2},\ 48)$.

Since this can be done in three ways, it is clear why the cube root enters into the expression of the coefficients of the canonical development.

The vertices of this triangle give a fundamental system of covariants, which is valid whenever $\Theta_3 \neq 0$.

The canonical development is identical for two differential equations whose absolute invariants are identical. We see, therefore, that *the condition, that two differential equations have equal absolute invariants, is not only necessary but also sufficient for their equivalence.*

§ 8. Anharmonic curves.

A curve is said to be anharmonic, if the absolute invariant $\Theta_8{}^3 : \Theta_3{}^8$ is a constant. Let us reduce the differential equation of the curve to the *Halphen* canonical form, which may always be done unless the curve is a conic. Then we shall have

$$p_1 = 0, \quad \Theta_3 = 1, \quad \Theta_8 = c,$$

where c is a constant, so that

$$p_2 = P_2 = -\frac{1}{27}c, \quad p_3 = P_3 = 1.$$

The differential equation becomes

$$y^{(3)} - \frac{1}{9}cy' + y = 0, \tag{52}$$

an equation with constant coefficients.

It is evident, on the other hand that, for any linear differential equation with constant coefficients, $\Theta_8{}^3 : \Theta_3{}^8$ will be a constant. If such an equation be reduced to its semi-canonical form, the coefficients will still be constants. Let

(53) $$y^{(3)} + 3P_2 y' + P_3 y = 0$$

be such an equation in its semi-canonical form, so that P_2 and P_3 are constants. Let r_1, r_2, r_3 be the roots, supposed distinct, of the cubic equation

(54) $$r^3 + 3P_2 r + P_3 = 0,$$

so that

(55) $$r_1 + r_2 + r_3 = 0, \quad r_2 r_3 + r_3 r_1 + r_1 r_2 = 3P_2, \quad -r_1 r_2 r_3 = P_3.$$

Then the functions

(56) $$y_k = e^{r_k x}, \quad (k = 1, 2, 3)$$

form a fundamental system of solutions of (53). The homogeneous equation of the integral curve of (53) may therefore be written

(57a) $$y_2^{r_3 - r_1} y_1^{r_2 - r_3} = y_3^{r_2 - r_1},$$

or in non-homogeneous form

(57b) $$Y = X^\lambda,$$

if we put

(58) $$X = \frac{y_2}{y_1}, \quad Y = \frac{y_3}{y_1}, \quad \lambda = \frac{r_3 - r_1}{r_2 - r_1}.$$

The curve, therefore, admits a one-parameter group of projective transformations into itself, viz:

(59a) $$\bar{X} = aX \quad \bar{Y} = a^\lambda Y,$$

where a is an arbitrary constant, or in homogeneous form

(59b) $$\bar{y}_1 = y_1, \quad \bar{y}_2 = a y_2, \quad \bar{y}_3 = a^\lambda y_3.$$

This group clearly enables us to convert any point of the curve into any other, excepting those vertices of the triangle of reference which are points of the curve. This property of anharmonic curves, that they are projectively equivalent to themselves, is characteristic of them.[1])

We can deduce from this theorem a corollary which justifies the name which we have given to these curves. Consider any point P of the curve, not a vertex of the fundamental triangle, together with its tangent. The latter intersects the sides of the triangle in three points P_1, P_2, P_3. *The anharmonic ratio of these four points, the point of contact and the intersections with the sides of the triangle, is the same for all tangents of the curve.* In fact, a projective transformation of the form (59) converts P and its tangent into any other point Q of the curve and its tangent. The points P_1, P_2, P_3 are

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1) cf. *Lie-Scheffers, Continuierliche Gruppen*, p 68 et sequ where other properties of these curves are also investigated. The above property is due to *Klein* and *Lie*.

converted into the three points Q_1, Q_2, Q_3 where the latter tangent intersects the sides of the fundamental triangle, this triangle being invariant under the transformation. But a projective transformation never changes the anharmonic ratio of four collinear points, whence our conclusion. The same result may, of course, be obtained by calculating this anharmonic ratio. It comes out to be equal to λ, if the four points are arranged in the proper order. λ is clearly an absolute invariant of the curve, and it must be possible to express it in terms of the absolute invariant $\frac{\Theta_8^3}{\Theta_3^8}$.

From (55) and (58) we find

$$P_2 = \frac{1}{3}\left[r_2 r_3 - (r_2 + r_3)^2\right], \quad P_3 = r_2 r_3 (r_2 + r_3),$$

$$\lambda = \frac{r_2 + 2r_3}{2r_2 + r_3},$$

whence

$$P_2 = -r_2^2 \frac{\lambda^2 - \lambda + 1}{(\lambda - 2)^2}, \quad P_3 = -r_2^3 \frac{(1 - 2\lambda)(\lambda + 1)}{(\lambda - 2)^2}$$

On the other hand, we shall have,

$$\Theta_3 = P_3, \quad \Theta_8 = -27 P_2 P_3^2,$$

whence

$$\frac{\Theta_8^3}{\Theta_3^8} = 3^9 \frac{(\lambda^2 - \lambda + 1)^3}{(\lambda - 2)^2 (1 - 2\lambda)^2 (\lambda + 1)^2}, \tag{60}$$

the equation connecting the invariant $\Theta_8^3 : \Theta_3^8$ with the invariant λ For $\lambda = 3$ we find again, as we should, the condition (38) for a cuspidal cubic.

This equation might have been derived in another way, which makes its significance more apparent If the numerical value of the invariant $\Theta_8^3 : \Theta_3^8$ be given, the curve C_y must be determined except for projective transformations. We would obtain at once, therefore, either equation (57a) or one of those obtained from it by the permutation of the indices 1, 2, 3. Corresponding to one value of $\Theta_8^3 : \Theta_3^8$ we would find, therefore, six values of λ, corresponding to these permutations, which turn out moreover to be expressible in the same way as the six values of the double-ratio of four points. In fact we have seen that λ really is a double-ratio. To one value of λ, however, would correspond only one value of $\Theta_8^3 : \Theta_3^8$. The left member of (60) must therefore be equal to a rational function of λ of the sixth degree, which is not changed by any substitution which consists in replacing λ by any of the functions:

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda}{\lambda - 1}, \quad \frac{\lambda - 1}{\lambda}.$$

This determines the right member of (60) except for a constant factor. This factor may be determined by considering the special case $\lambda = 3$.

If (60) is verified, the general solution of (1) is of the form

$$Y = X^\lambda,$$

where X and Y are expressions of the form

$$X = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{c_1 y_1 + c_2 y_2 + c_3 y_3}, \quad Y = \frac{b_1 y_1 + b_2 y_2 + b_3 y_3}{c_1 y_1 + c_2 y_2 + c_3 y_3},$$

apparently containing eight arbitrary constants, the ratios of the nine coefficients. But there are really only seven arbitrary constants, owing to the fact which we have already noted, that there exists a one-parameter group of projective transformations which converts the curve into itself. In accordance with this, if we express (60) by means of *Halphen's* differential invariants, it becomes a differential equation of the 7^{th} order between X and Y.

We have assumed so far that r_1, r_2 and r_3 are distinct. If two of these quantities coincide, we find that λ assumes one of the values 0, 1 or ∞. Equation (60) retains its significance, and we may determine the character of the integral curve as follows. If λ has a finite value, a special solution may be taken of the form

$$Y = \left(1 + \frac{X}{\lambda}\right)^\lambda.$$

Then, since

$$\lim_{\lambda = \infty} \left(1 + \frac{X}{\lambda}\right)^\lambda = e^X,$$

we see that corresponding to $\lambda = \infty$, a special integral curve of (1) may be written in the form

$$Y = e^X, \text{ or } X = \log Y.$$

Since all integral curves of (1) are projectively equivalent, we have the following result. *The anharmonic curves corresponding to the case that two of the roots of the characteristic cubic equation* (54) *coincide, are obtained by putting λ equal to* 0, 1 *or* ∞. *Their general form is*

$$X = \log Y,$$

where

$$X = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{c_1 y_1 + c_2 y_2 + c_3 y_3}, \quad Y = \frac{b_1 y_1 + b_2 y_2 + b_3 y_3}{c_1 y_1 + c_2 y_2 + c_3 y_3}.$$

The one-parameter projective group of these curves assumes the form

$$\overline{X} = X + \log a, \quad \overline{Y} = aY.$$

If all three roots of the cubic (54) are equal, they must be zero, whence we find $P_2 = P_3 = 0$, and therefore $\Theta_3 = 0$. In this case the integral curve is a conic. The corresponding values of λ are

seen to be $\lambda = 2, \frac{1}{2}$ or -1, the harmonic values of a double ratio. We may therefore describe a conic as an harmonic anharmonic curve.

In the metrical theory of plane curves, the osculating circle serves to characterize the infinitesimal properties of a curve in the vicinity of a given point. The *osculating anharmonic curve* may serve a similar purpose in the projective theory. We have seen, in § 7, that any curve, which is not a conic, may have its equation written in the form

$$Y = \frac{1}{2} X^2 + X^5 + \frac{(-20)^{\frac{8}{8}}}{100\,800} \frac{\Theta_8}{\Theta_3^{\frac{8}{3}}} X^7 + \cdots,$$

so that no curve, except conics and straight lines, have any projective infinitesimal properties expressed by derivatives of Y of lower order than 7. In the language of *Halphen's* differential invariants, we would say that there exists no absolute differential invariant of order lower than seven. We may, however, construct an anharmonic curve whose absolute invariant shall coincide with that of the given curve at the given point. Then its development in the canonical form will coincide with the above, up to and including the terms of the seventh order. It has contact of the seventh order with C_{y} at P_y. Therefore, the osculating anharmonic gives an adequate representation of the differential invariant of lowest order. Its determination may be regarded as an approximate integration of the differential equation (1).

§ 9. Discussion of the special case $\Theta_3 = 0$.

The general theorems in regard to semi-covariants specialize into well-known properties of conic sections when $\Theta_3 = 0$. Take the equation in its canonical form

$$y^{(3)} = 0,$$

so that we may put

$$y_1 = 1, \quad y_2 = x, \quad y_3 = x^2, \qquad y_2^{\,2} - y_1 y_3 = 0,$$
$$z_1 = 0, \quad z_2 = 1, \quad z_3 = 2x,$$
$$\varrho_1 = 0, \quad \varrho_2 = 0, \quad \varrho_3 = 2.$$

The curve C_ϱ degenerates, therefore, into a point on the conic; C_z is its tangent. From the general theorems we deduce the well known property that a moving tangent intersects four fixed tangents of the conic in a point-row of constant cross-ratio. We need not insist upon these matters.

The linear covariants, except y, vanish identically. The quadratic covariant C_2 however retains its significance. For the canonical form we have

$$C_2 = z^2 - 2y\varrho.$$

If we put for y the general solution of the differential equation

$$y = a + 2bx + cx^2,$$

and thence

$$z = 2b + 2cx, \quad \varrho = 2c,$$

we find

$$C_2 = 4\ (b^2 - ac),$$

the discriminant of the quadratic. Therefore, the three values of C_2 which correspond to the substitution of $y = y_k$ $(k = 1, 2, 3)$ are the discriminants of the quadratic equations which determine the intersection of the conic with the three sides of the triangle composed of two tangents and the polar of their intersection.

Examples.

Ex. 1. Assuming that the differential equation of C_y is written in its semi-canonical form, find the differential equations for C_z and C_ϱ.

Ex. 2. Find and discuss the conditions that C_z, C_ϱ, C_α may be straight lines or conics.

Ex 3. Prove that the third tangential of P_y coincides with the *Halphen* point of P_y (cf. end of § 3). Find the conditions that the loci of the points P_h, P_γ, P_δ there mentioned may be straight lines or conics.

Ex. 4. Find the conditions for a curve of the third class, discussing the various special cases.

CHAPTER IV.

INVARIANTS AND COVARIANTS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS.

§ 1. Finite transformations of the dependent variables.

Consider the system of linear homogeneous differential equations

$$(1) \qquad y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} p_{ikl} y_k^{(l)} = 0, \quad (i = 1, 2, \ldots n),$$

where

$$y_k^{(l)} = \frac{d^l y_k}{dx^l},$$

and where the quantities p_{ikl} are functions of x. It has been shown, in Chapter I, that the most general point transformation, which converts this system into another of the same kind, is given by the equations

$$y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}(\xi)\eta_\lambda, \quad x = f(\xi),$$

where $\alpha_{k\lambda}$ and f are arbitrary functions of ξ, and where the determinant

$$|\alpha_{k\lambda}(\xi)| \quad (k, \lambda = 1, 2, \ldots n)$$

does not vanish identically.

A function of the coefficients of (1) and of their derivatives, which has the same value for (1) as for any system obtained from it by such a transformation, shall be called an *absolute invariant.* If it contains also the functions y_k, y_k', etc. it is called a *covariant*

As in the case of a single linear differential equation, we decompose the transformation into two others; one, affecting only the dependent variables, and one transforming the independent variable. We proceed to determine first those functions which remain invariant when the dependent variables alone are transformed We shall speak of them as *seminvariants* and *semi-covariants.* The invariants and covariants will be functions of the seminvariants and semi-covariants

We proceed, therefore, to transform (1) by putting

$$(2) \qquad y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}(x)\eta_\lambda, \quad (k = 1, 2, \ldots n).$$

Then

$$(3) \quad y_k^{(l)} = \sum_{\lambda=1}^{n} \sum_{\rho=0}^{l} \binom{l}{\rho} \alpha_{k\lambda}^{(\rho)} \eta_\lambda^{(l-\rho)}, \quad (k = 1, 2, \ldots n;\ l = 0, 1, 2 \ldots m),$$

where $\binom{l}{\rho}$ denotes the coefficient of x^ρ in the expansion of $(1+x)^l$

Equations (1) become

$$(4) \quad \sum_{\lambda=1}^{n} \alpha_{i\lambda}\eta_\lambda^{(m)} + \sum_{\lambda=1}^{n} \sum_{\rho=1}^{m} \binom{m}{\rho} \alpha_{i\lambda}^{(\rho)} \eta_\lambda^{(m-\rho)}$$
$$+ \sum_{l=0}^{m-1} \sum_{k=1}^{n} \sum_{\mu=1}^{n} \sum_{\sigma=0}^{l} \binom{l}{\sigma} p_{ikl} \alpha_{k\mu}^{(\sigma)} \eta_\mu^{(l-\sigma)} = 0, \quad (i = 1, 2, \ldots n)$$

The coefficient of $\eta_\mu^{(\nu)}$ in the double sum is

$$\binom{m}{m-\nu} \alpha_{i\mu}^{(m-\nu)};$$

in the quadruple sum, the coefficient of $\eta_\mu^{(\nu)}$ is

$$\sum_{k=1}^{n} \left[\binom{\nu}{0} p_{ik\nu} \alpha_{k\mu} + \binom{\nu+1}{1} p_{i,k,\nu+1} \alpha_{k\mu}^{(1)} + \cdots \right.$$
$$\left. + \binom{m-1}{m-1-\nu} p_{i,k,m-1} \alpha_{k\mu}^{(m-1-\nu)} \right],$$

or

$$\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{\iota,k,\nu+\tau}\,\alpha_{k\mu}^{(\tau)}.$$

Thus, equations (4) may be written

$$(5)\qquad \sum_{\lambda=1}^{n}\alpha_{\iota\lambda}\eta_{\lambda}^{(m)}+\sum_{\mu=1}^{n}\sum_{\nu=0}^{m-1}\eta_{\mu}^{(\nu)}\left[\binom{m}{m-\nu}\alpha_{\iota\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{\iota,k,\nu+\tau}\,\alpha_{k\mu}^{(\tau)}\right]=0,\ (\iota=1,2,\dots n)$$

Put

$$(6)\qquad \Delta=|\alpha_{\iota\lambda}|\quad(\iota,\lambda=1,2,\dots n),$$

and denote by $A_{\iota\lambda}$ the minor of $\alpha_{\iota\lambda}$ in this determinant. Then we shall find

$$(7)\qquad \Delta\eta_{\lambda}^{(m)}+\sum_{\mu=1}^{n}\sum_{\nu=0}^{m-1}\eta_{\mu}^{(\nu)}\sum_{\iota=1}^{n}A_{\iota\lambda}\left[\binom{m}{m-\nu}\alpha_{\iota\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{\iota,k,\nu+\tau}\,\alpha_{k\mu}^{(\tau)}\right]=0,\ (\lambda=1,2,\dots n).$$

If the system be written in the form

$$(8)\qquad \eta_{\lambda}^{(m)}+\sum_{\nu=0}^{m-1}\sum_{\mu=1}^{n}\pi_{\lambda\mu\nu}\eta_{\mu}^{(\nu)}=0,$$

we shall, therefore, have

$$(9)\qquad \Delta\pi_{\lambda\mu\nu}=\sum_{\iota=1}^{n}A_{\iota\lambda}\left[\binom{m}{m-\nu}\alpha_{\iota\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{\iota,k,\nu+\tau}\,\alpha_{k\mu}^{(\tau)}\right]$$

$$(\lambda,\mu=1,2,\dots n;\ \nu=0,1,\dots m-1).$$

Thus, if (1) be transformed into (8) by means of the transformation (2), the equations (9) furnish the expressions for the new coefficients in terms of the old.

Equations (9) represent an infinite continuous group, isomorphic with the group represented by equations (2). For, they obviously have the group property, and to every transformation of the group (2) corresponds one of the group (9). Both groups can be defined by differential equations, so that *Lie's* theory of infinite groups may be applied.

§ 2. **Infinitesimal transformations of the dependent variables.**

The variables $y_1, y_2, \dots y_n$ will undergo the most general infinitesimal transformation of the form (2), if we put

(10) $$\alpha_{ii} = 1 + \varphi_{ii}(x)\,\delta t, \quad \alpha_{ik}(x) = \varphi_{ik}(x)\,\delta t, \quad (i \neq k, \quad i, k = 1, 2, \dots n),$$

where δt is an infinitesimal, and the φ_{ik}'s are arbitrary functions of x. We wish to find the corresponding infinitesimal transformations of the coefficients p_{ikl}.

Neglecting infinitesimals of order higher than the first, we find

(11) $$\Delta = \begin{vmatrix} 1 + \varphi_{11}\delta t, & \varphi_{12}\delta t, & \dots & \varphi_{1n}\delta t \\ \varphi_{21}\delta t, & 1 + \varphi_{22}\delta t, & \dots & \varphi_{2n}\delta t \\ \cdot & \cdot & & \cdot \\ \varphi_{n1}\delta t, & \varphi_{n2}\delta t, & \dots & 1 + \varphi_{nn}\delta t \end{vmatrix} = 1 + (\varphi_{11} + \varphi_{22} + \cdots + \varphi_{nn})\delta t,$$

and

(12) $$\begin{aligned} A_{ii} &= 1 + (\varphi_{11} + \varphi_{22} + \cdots + \varphi_{nn} - \varphi_{ii})\delta t, \\ A_{ki} &= -\varphi_{ik}\delta t, \quad (i \neq k) \end{aligned}$$

These latter formulae may be deduced from the equations

$$\sum_{k=1}^{n} \alpha_{ik} A_{ik} = \Delta, \quad \sum_{k=1}^{n} \alpha_{ik} A_{jk} = 0, \quad i \neq j,$$

which define the minors of the determinant Δ.

Substituting these values into (9), we have

$$\begin{aligned} \Delta \pi_{\lambda\mu\nu} = \sum_{i=1}^{n} -\varphi_{\lambda i}\delta t \Big[\binom{m}{m-\nu} \varphi_{i\mu}^{(m-\nu)} \delta t \\ + \sum_{k=1}^{n} \sum_{\tau=0}^{m-1-\nu} \binom{\nu+\tau}{\tau} p_{ik,\nu+\tau} \varphi_{k\mu}^{(\tau)} \delta t + p_{i\mu\nu} \Big] \\ + [1 + (\varphi_{11} + \varphi_{22} + \cdots + \varphi_{nn})\delta t] \Big[\binom{m}{m-\nu} \varphi_{\lambda\mu}^{(m-\nu)} \delta t \\ + \sum_{k=1}^{n} \sum_{\tau=0}^{m-1-\nu} \binom{\nu+\tau}{\tau} p_{\lambda k,\nu+\tau} \varphi_{k\mu}^{(\tau)} \delta t + p_{\lambda\mu\nu} \Big], \end{aligned}$$

or, omitting terms of higher than the first order in δt,

$$\begin{aligned} \Delta \pi_{\lambda\mu\nu} = p_{\lambda\mu\nu} - \sum_{i=1}^{n} \varphi_{\lambda i} p_{i\mu\nu} \delta t + \binom{m}{m-\nu} \varphi_{\lambda\mu}^{(m-\nu)} \delta t \\ + \sum_{k=1}^{n} \sum_{\tau=0}^{m-1-\nu} \binom{\nu+\tau}{\tau} p_{\lambda k,\nu+\tau} \varphi_{k\mu}^{(\tau)} \delta t + p_{\lambda\mu\nu}(\varphi_{11} + \varphi_{22} + \cdots + \varphi_{nn})\delta t. \end{aligned}$$

Dividing by $\Delta = 1 + (\varphi_{11} + \cdots + \varphi_{nn})\,\delta t$, and denoting the infinitesimal difference $\pi_{\lambda\mu\nu} - p_{\lambda\mu\nu}$ by $\delta p_{\lambda\mu\nu}$, we find

$$(13)\qquad \frac{\delta p_{\lambda\mu\nu}}{\delta t}=\sum_{k=1}^{n}(\varphi_{k\mu}p_{\lambda k\nu}-\varphi_{\lambda k}p_{k\mu\nu})$$
$$+\sum_{k=1}^{n}\sum_{\tau=1}^{m-1-\nu}\binom{\nu+\tau}{\tau}\varphi^{(\tau)}_{k\mu}p_{\lambda,k,\nu+\tau}+\binom{m}{m-\nu}\varphi^{(m-\nu)}_{\lambda\mu},$$
$$(\lambda,\mu=1,2,\ldots n;\ \nu=0,1,2,\ldots m-1).$$

These are the required infinitesimal transformations of $p_{\lambda\mu\nu}$. Those of $p'_{\lambda\mu\nu}$, $p''_{\lambda\mu\nu}$, etc., may be obtained from (13) by differentiation.

§ 3. Calculation of the seminvariants for $m=n=2$.

We proceed to the special case $m=n=2$ to which we shall confine ourselves. We may write our system of differential equations in the form

$$(14)\qquad \begin{aligned} &y''+p_{11}y'+p_{12}z'+q_{11}y+q_{12}z=0,\\ &z''+p_{21}y'+p_{22}z'+q_{21}y+q_{22}z=0,\end{aligned}$$

where we have written y and z in place of y_1 and y_2 as this notation will be more convenient later. We shall have to put in our general formulae:

$$(15)\qquad p_{\lambda,\mu,m-1}=p_{\lambda\mu1}=p_{\lambda\mu},\quad p_{\lambda,\mu,m-2}=p_{\lambda\mu0}=q_{\lambda\mu}.$$

Equations (13) will therefore become

$$(16)\qquad \begin{aligned} \frac{\delta p_{\lambda\mu}}{\delta t}&=\sum_{k=1}^{2}(\varphi_{k\mu}p_{\lambda k}-\varphi_{\lambda k}p_{k\mu})+2\varphi'_{\lambda\mu},\\ \frac{\delta p'_{\lambda\mu}}{\delta t}&=\sum_{k=1}^{2}(\varphi_{k\mu}p'_{\lambda k}-\varphi_{\lambda k}p'_{k\mu}+\varphi'_{k\mu}p_{\lambda k}-\varphi'_{\lambda k}p_{k\mu})+2\varphi^{(2)}_{\lambda\mu},\\ \frac{\delta q_{\lambda\mu}}{\delta t}&=\sum_{k=1}^{2}(\varphi_{k\mu}q_{\lambda k}-\varphi_{\lambda k}q_{k\mu})+\sum_{k=1}^{2}\varphi'_{k\mu}p_{\lambda k}+\varphi^{(2)}_{\lambda\mu}.\end{aligned}$$

If f is a seminvariant depending only upon the arguments $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$ the expression

$$\delta f=\sum_{\lambda,\mu}\left(\frac{\partial f}{\partial p_{\lambda\mu}}\delta p_{\lambda\mu}+\frac{\partial f}{\partial p'_{\lambda\mu}}\delta p'_{\lambda\mu}+\frac{\partial f}{\partial q_{\lambda\mu}}\delta q_{\lambda\mu}\right),$$

which represents the increment which the infinitesimal transformation gives to f, must vanish for all values of the arbitrary functions φ_{rs}, φ_{rs}', φ_{rs}''. Equating to zero the coefficients of these twelve arbitrary functions in δf, gives a system of partial differential equations for f. The general theory teaches us that it is a complete system, and that any solution of it is a seminvariant.

For abbreviation let us put always

$$(17) \qquad \frac{\partial f}{\partial p_{\lambda\mu}} = P_{\lambda\mu}, \quad \frac{\partial f}{\partial p'_{\lambda\mu}} = P'_{\lambda\mu}, \quad \frac{\partial f}{\partial q_{\lambda\mu}} = Q_{\lambda\mu}, \text{ etc.}$$

Then we have, for the seminvariants depending only upon $p_{\lambda\mu}$, $p'_{\lambda\mu}$ and $q_{\lambda\mu}$, the following complete system of partial differential equations:

$$(18) \qquad \begin{aligned} &2P'_{rs} + Q_{rs} = 0, \\ &2P_{rs} + \sum_{\lambda=1}^{2} (p_{\lambda r}P'_{\lambda s} - p_{s\lambda}P'_{r\lambda} + p_{\lambda r}Q_{\lambda s}) = 0, \\ &\sum_{\lambda=1}^{2} (p_{\lambda r}P_{\lambda s} - p_{s\lambda}P_{r\lambda} + p'_{\lambda r}P'_{\lambda s} - p'_{s\lambda}P'_{r\lambda} + q_{\lambda r}Q_{\lambda s} - q_{s\lambda}Q_{r\lambda}) = 0, \\ &\qquad\qquad (r, s = 1, 2). \end{aligned}$$

This contains twelve equations with twelve independent variables. But we shall see that only ten of the equations are independent, so that there are two seminvariants containing only the variables $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$.

The first four equations of (18) tell us that p'_{rs} and q_{rs} can occur only in the combinations

$$p'_{rs} - 2q_{rs}.$$

We shall write out the next four equations explicitly. They are:

$$(19) \qquad \begin{aligned} &2P_{12} + (p_{11} - p_{22})P'_{12} + p_{21}(P'_{22} - P'_{11}) + p_{11}Q_{12} + p_{21}Q_{22} = 0, \\ &2P_{11} + p_{21}P'_{21} - p_{12}P'_{12} + p_{11}Q_{11} + p_{21}Q_{21} = 0, \\ &2P_{22} + p_{12}P'_{12} - p_{21}P'_{21} + p_{12}Q_{12} + p_{22}Q_{22} = 0, \\ &2P_{21} + (p_{22} - p_{11})P'_{21} + p_{12}(P'_{11} - P'_{22}) + p_{12}Q_{11} + p_{22}Q_{21} = 0. \end{aligned}$$

They show that p_{rs}, p'_{rs}, q_{rs} can occur only in the four combinations:

$$(20) \qquad \begin{aligned} u_{11} &= 2p'_{11} - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} &= 2p'_{12} - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} &= 2p'_{21} - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} &= 2p'_{22} - 4q_{22} + p_{22}^2 + p_{12}p_{21}; \end{aligned}$$

i. e. the seminvariants here considered must be functions of $u_{11}, u_{12}, u_{21}, u_{22}$.

Finally we shall write out the last four equations of (18). They are:

$$(21) \qquad \begin{aligned} U_1 = {} & (p_{11} - p_{22})P_{12} + p_{21}(P_{22} - P_{11}) + (p'_{11} - p'_{22})P'_{12} \\ & + p'_{21}(P'_{22} - P'_{11}) + (q_{11} - q_{22})Q_{12} + q_{21}(Q_{22} - Q_{11}) = 0, \\ U_2 = {} & p_{21}P_{21} - p_{12}P_{12} + p'_{21}P'_{21} - p'_{12}P'_{12} + q_{21}Q_{21} - q_{12}Q_{12} = 0, \\ U_3 = {} & p_{12}P_{12} - p_{21}P_{21} + p'_{12}P'_{12} - p'_{21}P'_{21} + q_{12}Q_{12} - q_{21}Q_{21} = 0, \\ U_4 = {} & (p_{22} - p_{11})P_{21} + p_{12}(P_{11} - P_{22}) + (p'_{22} - p'_{11})P'_{21} \\ & + p'_{12}(P'_{11} - P'_{22}) + (q_{22} - q_{11})Q_{21} + q_{12}(Q_{11} - Q_{22}) = 0, \end{aligned}$$

with the obvious relation

$$(22) \qquad U_2 + U_3 = 0.$$

We find

$$U_1(u_{11}) = -u_{21}, \quad U_2(u_{11}) = 0, \quad U_3(u_{11}) = 0, \quad U_4(u_{11}) = +u_{12},$$
$$U_1(u_{12}) = u_{11} - u_{22}, \quad U_2(u_{12}) = -u_{12}, \quad U_3(u_{12}) = +u_{12}, \quad U_4(u_{12}) = 0,$$
$$U_1(u_{21}) = 0, \quad U_2(u_{21}) = +u_{21}, \quad U_3(u_{21}) = -u_{21}, \quad U_4(u_{21}) = u_{22} - u_{11},$$
$$U_1(u_{22}) = +u_{21}, \quad U_2(u_{22}) = 0, \quad U_3(u_{22}) = 0, \quad U_4(u_{22}) = -u_{12}.$$

With the introduction of these variables (21) becomes

$$\begin{aligned}
&U_1 f = -u_{21}\frac{\partial f}{\partial u_{11}} + (u_{11} - u_{22})\frac{\partial f}{\partial u_{12}} + u_{21}\frac{\partial f}{\partial u_{22}} = 0,\\
(23)\qquad &U_4 f = +u_{12}\frac{\partial f}{\partial u_{11}} - (u_{11} - u_{22})\frac{\partial f}{\partial u_{21}} - u_{12}\frac{\partial f}{\partial u_{22}} = 0,\\
&U_2 f = -u_{12}\frac{\partial f}{\partial u_{12}} + u_{21}\frac{\partial f}{\partial u_{21}} = 0,
\end{aligned}$$

where the relation

$$(23\text{a})\qquad u_{12} U_1 f + u_{21} U_4 f + (u_{11} - u_{22}) U_2 f = 0$$

is fulfilled, so that (23) will have two solutions. There will be, therefore, two seminvariants depending upon the variables p_{rs}, p'_{rs} and q_{rs}, viz.:

$$(24)\qquad I = u_{11} + u_{22}, \quad J = u_{11}u_{22} - u_{12}u_{21}.$$

Let us proceed to obtain next those seminvariants which contain also the quantities p''_{ik} and q'_{ik}. They must satisfy the following system of partial differential equations:

$$\begin{aligned}
&(\text{a})\quad 2P''_{rs} + Q'_{rs} = 0,\\
&(\text{b})\quad 2P'_{rs} + Q_{rs} + \sum_{\lambda=1}^{2}(p_{\lambda r}P''_{\lambda s} - p_{s\lambda}P''_{r\lambda} + p_{\lambda r}Q'_{\lambda s}) = 0,\\
(25)\quad &(\text{c})\quad 2P_{rs} + \sum_{\lambda=1}^{2}(p_{\lambda r}P'_{\lambda s} - p_{s\lambda}P'_{r\lambda} - 2p'_{s\lambda}P''_{r\lambda}\\
&\qquad\qquad + q_{\lambda r}Q'_{\lambda s} - q_{s\lambda}Q'_{r\lambda} + p_{\lambda r}Q_{\lambda s}) = 0,\\
&(\text{d})\quad \sum_{\lambda=1}^{2}(p_{\lambda r}P_{\lambda s} - p_{s\lambda}P_{r\lambda} + p'_{\lambda r}P'_{\lambda s} - p'_{s\lambda}P'_{r\lambda} + p''_{\lambda r}P''_{\lambda s}\\
&\qquad - p''_{s\lambda}P''_{r\lambda} + q_{\lambda r}Q_{\lambda s} - q_{s\lambda}Q_{r\lambda} + q'_{\lambda r}Q'_{\lambda s} - q'_{s\lambda}Q'_{r\lambda}) = 0,\\
&\qquad\qquad (r,\ s = 1,\ 2).
\end{aligned}$$

There are in this system 20 independent variables, and 16 equations. Only 15 of these equations are independent, there being a relation between the equations (d) which reduces to $U_2 + U_3 = 0$ when f contains only p_{rs}, p'_{rs} and q_{rs}. In fact the left members of (d) for $r = 1$, $s = 2$ and for $r = 2$, $s = 1$ differ only by the factor -1. We shall see that these 15 equations are actually independent; the other relation which was found in the previous case does not main-

tain itself. There must, therefore, be five seminvariants of the kind considered. Of these we know four, namely I, J, $\frac{dI}{dx}$, $\frac{dJ}{dx}$, which are obviously independent. It remains to find the fifth.

Put

$$u'_{ik} = \frac{du_{ik}}{dx}, \text{ etc.}$$

Then, since according to (20) we have

$$(26) \qquad u_{ik} = 2p'_{ik} - 4q_{ik} + \sum_{j=1}^{2} p_{ij}p_{jk},$$

therefore

$$(27) \qquad u'_{ik} = 2p''_{ik} - 4q'_{ik} + \sum_{j=1}^{2} (p_{ij}p'_{jk} + p'_{ij}p_{jk}).$$

It is easy to show from equations (25) (a) and (b) that our seminvariants are functions of the twelve arguments

$$(28) \qquad p_{ik}, \quad u_{ik}, \quad u'_{ik}.$$

Denote the left members of (25) (c) by $\Omega_1, \cdots \Omega_4$, so that $\Omega_1 = 2P_{11} + \cdots$, $\Omega_2 = 2P_{12} + \cdots$, $\Omega_3 = 2P_{21} + \cdots$, $\Omega_4 = 2P_{22} + \cdots$

Then we find:

$$(29) \qquad \Omega_1(u_{ik}) = \Omega_2(u_{ik}) = \Omega_3(u_{ik}) = \Omega_4(u_{ik}) = 0;$$

further

$$(30) \qquad \begin{array}{llll} \Omega_1(p_{11}) = 2, & \Omega_2(p_{11}) = 0, & \Omega_3(p_{11}) = 0, & \Omega_4(p_{11}) = 0, \\ \Omega_1(p_{12}) = 0, & \Omega_2(p_{12}) = 2, & \Omega_3(p_{12}) = 0, & \Omega_4(p_{12}) = 0, \\ \Omega_1(p_{21}) = 0, & \Omega_2(p_{21}) = 0, & \Omega_3(p_{21}) = 2, & \Omega_4(p_{21}) = 0, \\ \Omega_1(p_{22}) = 0, & \Omega_2(p_{22}) = 0, & \Omega_3(p_{22}) = 0, & \Omega_4(p_{22}) = 2; \end{array}$$

and finally:

$$(31) \qquad \begin{array}{lll} \Omega_1(u'_{11}) = 0, & \Omega_2(u'_{11}) = -u_{21}, & \\ & \Omega_3(u'_{11}) = +u_{12}, & \Omega_4(u'_{11}) = 0, \\ \Omega_1(u'_{12}) = -u_{12}, & \Omega_2(u'_{12}) = u_{11} - u_{22}, & \\ & \Omega_3(u'_{12}) = 0, & \Omega_4(u'_{12}) = +u_{12}, \\ \Omega_1(u'_{21}) = +u_{21}, & \Omega_2(u'_{21}) = 0, & \\ & \Omega_3(u'_{21}) = -(u_{11} - u_{22}), & \Omega_4(u'_{21}) = -u_{21}, \\ \Omega_1(u'_{22}) = 0 & \Omega_2(u'_{22}) = +u_{21} & \\ & \Omega_3(u'_{22}) = -u_{12}, & \Omega_4(u'_{22}) = 0. \end{array}$$

From these equations it is easily seen that the eight independent functions of the arguments (28), which verify the equations $\Omega_\lambda = 0$, are the quantities u_{ik} and v_{ik}, where

$$
(32) \qquad \begin{aligned}
v_{11} &= 2u'_{11} + p_{12}u_{21} - p_{21}u_{12} \\
v_{12} &= 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\
v_{21} &= 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\
v_{22} &= 2u'_{22} - p_{12}u_{21} + p_{21}u_{12}.
\end{aligned}
$$

Denoting by $X_1, \ldots X_4$ the left members of (25) (d) we find:

$$
(33) \qquad \begin{array}{llll}
X_1(v_{11}) = 0, & X_2(v_{11}) = -v_{21}, & X_3(v_{11}) = +v_{12}, & X_4(v_{11}) = 0. \\
X_1(v_{12}) = -v_{12} & X_2(v_{12}) = v_{11} - v_{22}, & X_3(v_{12}) = 0, & X_4(v_{12}) = +v_{12}, \\
X_1(v_{21}) = +v_{21} & X_2(v_{21}) = 0, & X_3(v_{21}) = -(v_{11} - v_{22}), & X_4(v_{21}) = -v_{21}, \\
X_1(v_{22}) = 0, & X_2(v_{22}) = +v_{21}, & X_3(v_{22}) = -v_{12} & X_4(v_{22}) = 0,
\end{array}
$$

the equations for $X_\lambda\ (u_{ik})$ being of precisely the same form. We have of course

$$X_1 + X_4 = 0,$$

the one relation between the sixteen partial differential equations.

If the variables u_{ik} and v_{ik} are introduced as independent variables, our system of equations $X_\lambda f = 0$ becomes therefore:

$$
\begin{aligned}
X_1 f &= \quad v_{12}\frac{\partial f}{\partial v_{12}} + v_{21}\frac{\partial f}{\partial v_{21}} - u_{12}\frac{\partial f}{\partial u_{12}} + u_{21}\frac{\partial f}{\partial u_{21}} = 0, \\
X_2 f &= -v_{21}\left(\frac{\partial f}{\partial v_{11}} - \frac{\partial f}{\partial v_{22}}\right) + (v_{11} - v_{22})\frac{\partial f}{\partial v_{12}} - u_{21}\left(\frac{\partial f}{\partial u_{11}} - \frac{\partial f}{\partial u_{22}}\right) \\
&\qquad + (u_{11} - u_{22})\frac{\partial f}{\partial u_{12}} = 0, \\
X_3 f &= +v_{12}\left(\frac{\partial f}{\partial v_{11}} - \frac{\partial f}{\partial v_{22}}\right) - (v_{11} - v_{22})\frac{\partial f}{\partial v_{21}} + u_{12}\left(\frac{\partial f}{\partial u_{11}} - \frac{\partial f}{\partial u_{22}}\right) \\
&\qquad - (u_{11} - u_{22})\frac{\partial f}{\partial u_{21}} = 0,
\end{aligned}
$$

which three equations are obviously independent; in the case that f is independent of $v_{11} \ldots v_{22}$ we find, of course, the same relation between the left members (23a) as before, only the notation being changed.

By integrating this system, or more simply from (33), we see that

$$v_{11} + v_{22} \text{ and } v_{11}v_{22} - v_{12}v_{21}$$

are solutions of the equations $X_\lambda = 0$. But

$$(34) \qquad v_{11} + v_{22} = 2I',$$

while

(35) $$K = v_{11}v_{22} - v_{12}v_{21}$$

is obviously a new seminvariant, independent of I, J, I', J'.

We might now write down the differential equations satisfied by the seminvariants involving, besides the quantities already considered, also $p_{ik}^{(3)}$ and q_{ik}''. We should find a system of twenty such equations with one relation between them, and twenty-eight independent variables. Hence there must be $28 - 19 = 9$ such seminvariants. But we know eight of these, viz.:

(36) $$I,\ I',\ I'';\ J,\ J',\ J'';\ K,\ K';$$

these are independent, for it is easily seen that from the existence of a relation between them would follow the existence of a relation between I, I', J, J', K. But these quantities are independent.

We may obtain the ninth seminvariant without writing down and integrating the last mentioned system of twenty equations. The process which we shall employ is much more instructive, and has the further merit that it is capable of generalization to cases other than that here considered of $m = n = 2$.

We have remarked in connection with (33) that the expressions $X_\lambda(u_{ik})$ and $X_\lambda(v_{ik})$ are of precisely the same form. *We may express this by saying that the quantities u_{ik} and v_{ik} are cogredient.*

To make this more evident we may compute δu_{ik} and δv_{ik}. We find from (16) and (20)

(37) $$\begin{aligned} \frac{\delta u_{11}}{\delta t} &= \varphi_{21}u_{12} - \varphi_{12}u_{21}, \\ \frac{\delta u_{12}}{\delta t} &= (\varphi_{22} - \varphi_{11})\,u_{12} + \varphi_{12}\,(u_{11} - u_{22}), \\ \frac{\delta u_{21}}{\delta t} &= (\varphi_{11} - \varphi_{22})\,u_{21} + \varphi_{21}\,(u_{22} - u_{11}), \\ \frac{\delta u_{22}}{\delta t} &= -\varphi_{21}u_{12} + \varphi_{12}u_{21}, \end{aligned}$$

and from (16), (32) and (37),

(38) $$\begin{aligned} \frac{\delta v_{11}}{\delta t} &= \varphi_{21}v_{12} - \varphi_{12}v_{21}, \\ \frac{\delta v_{12}}{\delta t} &= (\varphi_{22} - \varphi_{11})\,v_{12} + \varphi_{12}\,(v_{11} - v_{22}), \\ \frac{\delta v_{21}}{\delta t} &= (\varphi_{11} - \varphi_{22})\,v_{21} + \varphi_{21}\,(v_{22} - v_{11}), \\ \frac{\delta v_{22}}{\delta t} &= \varphi_{21}v_{12} + \varphi_{12}v_{21}. \end{aligned}$$

Now certain combinations of the u_{ik}'s and p_{ik}'s, namely $v_{11} + v_{22}$ and $v_{11}v_{22} - v_{12}v_{21}$ are seminvariants Since the v_{ik}'s are cogredient with the u_{ik}'s, the same combinations with v_{ik} in place of u_{ik} will also be seminvariants.

Let us, therefore, put

$$(39)\qquad \begin{aligned} w_{11} &= 2v'_{11} + p_{12}v_{21} - p_{21}v_{12}, \\ w_{12} &= 2v'_{12} + (p_{11} - p_{22})\,v_{12} - p_{12}\,(v_{11} - v_{22}), \\ w_{21} &= 2v'_{21} - (p_{11} - p_{22})\,v_{21} + p_{21}\,(v_{11} - v_{22}), \\ w_{22} &= 2v'_{22} - p_{12}v_{21} + p_{21}v_{12}, \end{aligned}$$

so that *the quantities* w_{ik} *are formed from* v_{ik} *and* p_{ik} *in the same way as the quantities* v_{ik} *are formed from* u_{ik} *and* p_{ik}. Of course w_{ik} are cogredient with u_{ik} and v_{ik}, so that $w_{11} + w_{22}$ and $w_{11}w_{22} - w_{12}w_{21}$ are seminvariants. But

$$w_{11} + w_{22} = 2(v'_{11} + v'_{22}) = 4I'',$$

while

$$(40)\qquad L = w_{11}w_{22} - w_{12}w_{21}$$

is a new seminvariant. That it is independent of the other eight seminvariants (36) may be verified by considering the special case $p_{ik} = 0$.

Our object, to find all of the seminvariants, is now accomplished. They are I, J, K, L and the derivatives of these quantities. For, suppose we wish to find the seminvariants involving the variables considered so far, and $p_{ik}^{(4)}$ and $q_{ik}^{(3)}$ besides. They are determined by a complete system of $24 - 1 = 23$ independent equations with 36 independent variables. Therefore, there exist $36 - 23 = 13$ such seminvariants. But as these we may take the nine which we have already found, together with $I^{(3)}$, $J^{(3)}$, K'', L'. These are certainly independent. Proceeding in this way, each step introduces eight new independent variables and four new equations. Each step, therefore, gives rise to four new seminvariants. But these four may clearly be obtained by performing an additional differentiation upon I, J, K, L.

Thus, all seminvariants, of the system of two linear homogeneous differential equations of the second order, are functions of the quantities I, J, K, L *and of their derivatives.*

It is interesting to note what would be the result of continuing our above process for obtaining seminvariants. Suppose we had formed

$$t_{11} = 2w'_{11} + p_{12}w_{21} - p_{21}w_{12}, \text{ etc.}$$

Then would

$$\begin{aligned} t_{11} - t_{22} &= g_1(u_{11} - u_{22}) + g_2(v_{11} - v_{22}) + g_3(w_{11} - w_{22}), \\ t_{12} &= g_1u_{12} + g_2v_{12} + g_3w_{12}, \\ t_{21} &= g_1u_{21} + g_2v_{21} + g_3w_{21}, \end{aligned}$$

where g_1, g_2, g_3 are seminvariants.

For g_1, g_2, g_3 are the quotients of determinants of the third order formed out of the matrix

$$\begin{vmatrix} t_{11} - t_{22}, & u_{11} & u_{22}, & v_{11} - v_{22}, & w_{11} - w_{22} \\ t_{12}, & & u_{12}, & v_{12}, & w_{12} \\ t_{21} & & u_{21}, & v_{21}, & w_{21} \end{vmatrix},$$

and on account of the cogredience of the four sets of quantities, such quotients are seminvariants. More than that, it may be verified at once that these determinants are themselves seminvariants. One of these is of special importance, viz.:

$$\Delta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix}. \tag{41}$$

We shall find later that Δ is also an invariant. Its expression in terms of I, J, K, L and of the derivatives of these quantities is not rational and will be given farther on. This remark suffices to show that the system of seminvariants, consisting of I, J, K, L and of their derivatives, is not complete in the sense that any rational seminvariant can be expressed as a *rational* function of them. Whether the system, obtained by adjoining Δ and its derivatives, is complete in this sense or not is a question which we shall leave open.

We shall frequently have occasion to make use of the finite transformations of p_{ik}, q_{ik}, u_{ik}, etc. These equations for p_{ik} and q_{ik} may be obtained at once from (9) by putting $m = n = 2$, or else directly. We prefer to take the transformation (2) in the form

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z}, \quad \alpha\delta - \beta\gamma = \Delta$$

so as to avoid the double indices. The coefficients, π_{ik} and $\varkappa_{ik}$, of the new system will then be given by the following equations:

$$\begin{aligned} \Delta\pi_{11} &= 2(\alpha'\delta - \gamma'\beta) + p_{11}\alpha\delta + p_{12}\gamma\delta - p_{21}\alpha\beta - p_{22}\gamma\beta, \\ \Delta\pi_{12} &= 2(\beta'\delta - \delta'\beta) + p_{11}\beta\delta + p_{12}\delta^2 - p_{21}\beta^2 - p_{22}\beta\delta, \\ \Delta\pi_{21} &= -2(\alpha'\gamma - \gamma'\alpha) - p_{11}\alpha\gamma - p_{12}\gamma^2 + p_{21}\alpha^2 + p_{22}\alpha\gamma, \\ \Delta\pi_{22} &= -2(\beta'\gamma - \delta'\alpha) - p_{11}\beta\gamma - p_{12}\gamma\delta + p_{21}\alpha\beta + p_{22}\alpha\delta, \end{aligned} \tag{42}$$

and

$$\begin{aligned} \Delta\varkappa_{11} &= \alpha''\delta - \gamma''\beta + p_{11}\alpha'\delta + p_{12}\gamma'\delta - p_{21}\alpha'\beta - p_{22}\gamma'\beta \\ &\qquad + q_{11}\alpha\delta + q_{12}\gamma\delta - q_{21}\alpha\beta - q_{22}\beta\gamma, \\ \Delta\varkappa_{12} &= \beta''\delta - \delta''\beta + p_{11}\beta'\delta + p_{12}\delta'\delta - p_{21}\beta'\beta - p_{22}\delta'\beta \\ &\qquad + q_{11}\beta\delta + q_{12}\delta^2 - q_{21}\beta^2 - q_{22}\beta\delta, \\ \Delta\varkappa_{21} &= -(\alpha''\gamma - \gamma''\alpha) - p_{11}\alpha'\gamma - p_{12}\gamma'\gamma + p_{21}\alpha'\alpha + p_{22}\gamma'\alpha \\ &\qquad - q_{11}\alpha\gamma - q_{12}\gamma^2 + q_{21}\alpha^2 + q_{22}\alpha\gamma, \\ \Delta\varkappa_{22} &= -(\beta''\gamma - \delta''\alpha) - p_{11}\beta'\gamma - p_{12}\delta'\gamma + p_{21}\beta'\alpha + p_{22}\delta'\alpha \\ &\qquad - q_{11}\beta\gamma - q_{12}\gamma\delta + q_{21}\alpha\beta + q_{22}\alpha\delta. \end{aligned} \tag{43}$$

The equations for π'_{ik} are obtained from (42) by differentiation. Denote by $\bar{u}_{ik}$ the values of u_{ik} for the new system of differential equations. We shall find

$$
(44) \qquad \begin{aligned}
\Delta \bar{u}_{11} &= \alpha\delta u_{11} + \gamma\delta u_{12} \quad \alpha\beta u_{21} - \beta\gamma u_{22}, \\
\Delta \bar{u}_{12} &= \beta\delta u_{11} + \delta^2 u_{12} - \beta^2 u_{21} - \beta\delta u_{22}, \\
\Delta \bar{u}_{21} &= -\alpha\gamma u_{11} - \gamma^2 u_{12} + \alpha^2 u_{21} + \alpha\gamma u_{22}, \\
\Delta u_{22} &= -\beta\gamma u_{11} - \gamma\delta u_{12} + \alpha\beta u_{21} + \alpha\delta u_{22}.
\end{aligned}
$$

This may be deduced from (42) and (43) without computation. For the equations (37) for the infinitesimal transformations show that $\bar{u}_{ik}$ must be a linear homogeneous function of $u_{11} \ldots u_{22}$, so that the terms in (42) and (43) which contain the derivatives of α, β, γ, δ must eliminate each other. Omitting these terms the quantities u_{ik} must be cogredient with q_{ik}, whence follows (44). The equations for v_{ik} and w_{ik} are, of course, of the same form as (44).

§ 4. Effect of a transformation of the independent variable upon the seminvariants.

The invariants of the system of linear differential equations must obviously be functions of the seminvariants, viz. such functions of the seminvariants as are left unchanged by an arbitrary transformation of the independent variable x. In order to determine them it becomes necessary to find the effect of such a transformation upon the seminvariants.

Let

$$
(45) \qquad \begin{aligned}
y'' + p_{11} y' + p_{12} z' + q_{11} y + q_{12} z &= 0, \\
z'' + p_{21} y' + p_{22} z' + q_{21} y + q_{22} z &= 0
\end{aligned}
$$

be the given system, as before Introduce a new independent variable

$$\xi = \xi(x)$$

Then (45) becomes

$$
\begin{aligned}
\frac{d^2 y}{d\xi^2} + \pi_{11} \frac{dy}{d\xi} + \pi_{12} \frac{dz}{d\xi} + \varkappa_{11} y + \varkappa_{12} z &= 0, \\
\frac{d^2 z}{d\xi^2} + \pi_{21} \frac{dy}{d\xi} + \pi_{22} \frac{dz}{d\xi} + \varkappa_{21} y + \varkappa_{22} z &= 0,
\end{aligned}
$$

where

$$
(46) \qquad \begin{gathered}
\pi_{11} = \frac{1}{\xi'}(p_{11} + \eta), \quad \pi_{12} = \frac{1}{\xi'} p_{12}, \quad \pi_{21} = \frac{1}{\xi'} p_{21}, \quad \pi_{22} = \frac{1}{\xi'}(p_{22} + \eta), \\
\varkappa_{ik} = \frac{1}{(\xi')^2} q_{ik}
\end{gathered}
$$

the quantity η being, as in previous chapters, defined by the equation:

$$
(47) \qquad \eta = \frac{\xi''}{\xi'}.
$$

We proceed to investigate the effect of this transformation upon u_{ik}, v_{ik} and w_{ik}. For the purpose of merely computing the invariants we might confine ourselves to the case of infinitesimal transformations. But, as we shall need to know the result of the general transformation for other purposes later on, we shall deduce the corresponding equations immediately.

We find, from (46),

$$(48)\quad \begin{aligned} \frac{d\pi_{11}}{d\xi} &= \frac{1}{(\xi')^2}[p'_{11} - \eta p_{11} + \eta' - \eta^2], & \frac{d\pi_{12}}{d\xi} &= \frac{1}{(\xi')^2}(p'_{12} - \eta p_{12}), \\ \frac{d\pi_{21}}{d\xi} &= \frac{1}{(\xi')^2}(p'_{21} - \eta p_{21}), & \frac{d\pi_{22}}{d\xi} &= \frac{1}{(\xi')^2}[p'_{22} - \eta p_{22} + \eta' - \eta^2], \end{aligned}$$

whence

$$(49)\quad \begin{aligned} \bar{u}_{11} &= \frac{1}{(\xi')^2}(u_{11} + 2\mu), & \bar{u}_{12} &= \frac{1}{(\xi')^2}u_{12}, \\ \bar{u}_{21} &= \frac{1}{(\xi')^2}u_{21}, & \bar{u}_{22} &= \frac{1}{(\xi')^2}(u_{22} + 2\mu), \end{aligned}$$

where

$$(50)\quad \mu = \eta' - \frac{1}{2}\eta^2 = \frac{\xi^{(3)}}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2$$

is the so-called *Schwarzian derivative*.

We find at once

$$(51)\quad \bar{I} = \frac{1}{(\xi')^2}(I + 4\mu), \quad \bar{J} = \frac{1}{(\xi')^4}(J + 2\mu I + 4\mu^2),$$

whence we may deduce an invariant

$$(52)\quad \Theta_4 = I^2 - 4J;$$

for we shall have

$$\bar{\Theta}_4 = \frac{1}{(\xi')^4}\Theta_4.$$

From the definitions of the quantities v_{ik} and w_{ik} we may now deduce the expressions for $\bar{v}_{ik}$ and $\bar{w}_{ik}$. We find

$$(53)\quad \begin{aligned} \bar{v}_{11} &= \frac{1}{(\xi')^3}(v_{11} - 4u_{11}\eta + 4\mu' - 8\mu\eta), & \bar{v}_{12} &= \frac{1}{(\xi')^3}(v_{12} - 4u_{12}\eta), \\ \bar{v}_{21} &= \frac{1}{(\xi')^3}(v_{21} - 4u_{21}\eta), & \bar{v}_{22} &= \frac{1}{(\xi')^3}(v_{22} - 4u_{22}\eta + 4\mu' - 8\mu\eta), \end{aligned}$$

where μ' denotes $\frac{d\mu}{dx}$. Further

$$(54)\quad \begin{aligned} \bar{w}_{11} &= \frac{1}{(\xi')^4}(w_{11} - 10v_{11}\eta + 20u_{11}\eta^2 - 8u_{11}\mu + 8\mu'' - 40\mu'\eta \\ &\qquad - 16\mu^2 + 40\mu\eta^2), \\ \bar{w}_{12} &= \frac{1}{(\xi')^4}(w_{12} - 10v_{12}\eta + 20u_{12}\eta^2 - 8u_{12}\mu), \\ \bar{w}_{21} &= \frac{1}{(\xi')^4}(w_{21} - 10v_{21}\eta + 20u_{21}\eta^2 - 8u_{21}\mu), \\ \bar{w}_{22} &= \frac{1}{(\xi')^4}(w_{22} - 10v_{22}\eta + 20u_{22}\eta^2 - 8u_{22}\mu + 8\mu'' - 40\mu'\eta \\ &\qquad - 16\mu^2 + 40\mu\eta^2). \end{aligned}$$

The most general infinitesimal transformation will be obtained by putting

(55) $$\xi(x) = x + \varphi(x)\delta t, \quad \delta x = \varphi(x)\delta t,$$

where $\varphi(x)$ is an arbitrary function and δt an infinitesimal. We shall then have

$$\xi' = 1 + \varphi'(x)\delta t, \quad \eta = \varphi''(x)\delta t, \quad \mu = \varphi^{(3)}(x)\delta t, \quad \mu' = \varphi^{(4)}(x)\delta t, \text{ etc.}\ldots,$$

whence, substituting in (46), we find

(56) $$\begin{aligned} \delta p_{11} &= (-\varphi' p_{11} + \varphi'')\delta t, & \delta p_{12} &= -\varphi' p_{12}\delta t, \\ \delta p_{21} &= -\varphi' p_{21}\delta t, & \delta p_{22} &= (-\varphi' p_{22} + \varphi'')\delta t, \end{aligned}$$
$$\delta q_{ik} = -2\varphi' q_{ik}\delta t, \quad (i, k = 1, 2).$$

Similarly we find from (49),

(57) $$\begin{aligned} \delta u_{11} &= (2\varphi^{(3)} - 2\varphi' u_{11})\delta t, & \delta u_{12} &= -2\varphi' u_{12}\delta t, \\ \delta u_{21} &= -2\varphi' u_{21}\delta t, & \delta u_{22} &= (2\varphi^{(3)} - 2\varphi u_{22})\delta t, \end{aligned}$$

whence

(58) $$\begin{aligned} \delta I &= (4\varphi^{(3)} - 2\varphi' I)\delta t, \\ \delta J &= (2\varphi^{(3)} I - 4\varphi' J)\delta t. \end{aligned}$$

Further

(59) $$\begin{aligned} \delta v_{11} &= (4\varphi^{(4)} - 4\varphi'' u_{11} - 3\varphi' v_{11})\delta t, \\ \delta v_{12} &= (-4\varphi'' u_{12} - 3\varphi' v_{12})\delta t, \\ \delta v_{21} &= (-4\varphi'' u_{21} - 3\varphi' v_{21})\delta t, \\ \delta v_{22} &= (4\varphi^{(4)} - 4\varphi'' u_{22} - 3\varphi' v_{22})\delta t, \end{aligned}$$

whence

(60) $$\delta K = (8\varphi^{(4)} I' - 8\varphi'' J' - 6\varphi' K)\delta t,$$

where the equation

(61) $$u_{11}v_{22} + u_{22}v_{11} - u_{12}v_{21} - u_{21}v_{12} = 2J'$$

has been used, the truth of which may be easily verified.

We find from (54),

(62) $$\begin{aligned} \delta w_{11} &= (8\varphi^{(5)} - 8\varphi^{(3)} u_{11} - 10\varphi'' v_{11} - 4\varphi' w_{11})\delta t. \\ \delta w_{12} &= (-8\varphi^{(3)} u_{12} - 10\varphi'' v_{12} - 4\varphi' w_{12})\delta t, \\ \delta w_{21} &= (-8\varphi^{(3)} u_{21} - 10\varphi'' v_{21} - 4\varphi' w_{21})\delta t, \\ \delta w_{22} &= (8\varphi^{(5)} - 8\varphi^{(3)} u_{22} - 10\varphi'' v_{22} - 4\varphi' w_{22})\delta t \end{aligned}$$

and notice the two equations similar to (61),

(63) $$\begin{aligned} v_{11}w_{22} + v_{22}w_{11} - v_{12}w_{21} - v_{21}w_{12} &= 2K', \\ w_{11}u_{22} + w_{22}u_{11} - w_{12}u_{21} - w_{21}u_{12} &= 2(2J'' - K). \end{aligned}$$

We shall then find:

(64) $$\delta L = [32\varphi^{(5)} I'' - 16\varphi^{(3)}(2J'' - K) - 20\varphi'' K' - 8\varphi' L]\delta t.$$

If f is any function of x and if $\bar{f}(\xi)$ denotes the result of substituting for x its value in terms of ξ, we have

$$\frac{d\bar{f}}{d\xi} = \frac{d\bar{f}}{dx} : \frac{d\xi}{dx}.$$

If the transformation is infinitesimal

$$\xi = x + \varphi(x)\delta t, \quad \bar{f} = f + \delta f,$$

so that

$$\frac{d\bar{f}}{d\xi} = \left[\frac{df}{dx} + \frac{d}{dx}(\delta f)\right]\left[1 - \varphi'\delta t\right],$$

or denoting $\frac{d\bar{f}}{d\xi} - \frac{df}{dx}$ by $\delta\left(\frac{df}{dx}\right)$, we find

$$\delta\left(\frac{df}{dx}\right) = \frac{d}{dx}(\delta f) - \varphi' \frac{df}{dx}\delta t. \tag{65}$$

By applying this formula we may easily find the infinitesimal transformation of the derivatives of I, J, K and L. We find

$$\begin{aligned}
&\delta I = (4\varphi^{(3)} - 2\varphi' I)\,\delta t,\\
(66)\qquad &\delta I' = (4\varphi^{(4)} - 2\varphi'' I - 3\varphi' I')\,\delta t,\\
&\delta I'' = (4\varphi^{(5)} - 2\varphi^{(3)} I - 5\varphi'' I' - 4\varphi' I'')\,\delta t.
\end{aligned}$$

Further, we shall have;

$$\begin{aligned}
&\delta J = (2\varphi^{(3)} I - 4\varphi' J)\,\delta t,\\
&\delta J' = (2\varphi^{(4)} I + 2\varphi^{(3)} I' - 4\varphi'' J - 5\varphi' J')\,\delta t,\\
(67)\quad &\delta J'' = [2\varphi^{(5)} I + 4\varphi^{(4)} I' + \varphi^{(3)}(2I'' - 4J) - 9\varphi'' J' - 6\varphi' J'']\,\delta t,\\
&\delta K = (8\varphi^{(4)} I' - 8\varphi'' J' - 6\varphi' K)\,\delta t,\\
&\delta K' = [8\varphi^{(5)} I' + 8\varphi^{(4)} I'' - 8\varphi^{(3)} J' - \varphi''(8J'' + 6K) - 7\varphi' K']\,\delta t.
\end{aligned}$$

These equations will be applied in the next paragraph.

§ 5. Calculation of some of the invariants. Their general properties.

Before proceeding to the calculation of the invariants and covariants, it becomes necessary to deduce certain general theorems corresponding to the general theorems of Chapter II

In the first place we may confine ourselves to covariants containing no higher derivatives of y and z than the first. For, by means of the fundamental differential equations all higher derivatives may be expressed in terms of y, z, y' and z'.

The function

$$C(y, z, y', z'; p_{ik}, p'_{ik}, \ldots; q_{ik}, q'_{ik}, \ldots),$$

which we shall assume to be an integral rational function of all of its arguments, not resolvable into rational factors, shall be called an

integral, rational, irreducible covariant if, for all transformations of the group G, the equation

$$C = 0$$

has as its consequence the equation

$$\Gamma = 0,$$

where Γ denotes the same function of the transformed quantities

$$\eta, \zeta, \ldots, \quad \pi_{ik}, \ldots, \quad \varkappa_{ik} \ldots,$$

as does C of the original variables The transformations of the group G are the transformations;

$$\xi = \xi(x), \quad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z, \quad \alpha\delta - \beta\gamma \neq 0$$

Let us make the transformation

$$\xi = x, \quad \eta = Cy, \quad \zeta = Cz, \quad C = \text{constant},$$

which belongs to the group G. We shall have (always denoting the transformed quantities by Greek letters),

$$\eta^{(\lambda)} = C y^{(\lambda)}, \quad \zeta^{(\lambda)} = C z^{(\lambda)}, \quad \pi_{ik} = p_{ik}, \quad \varkappa_{ik} = q_{ik}.$$

Therefore, *any covariant must be homogeneous in y, z, y', z'. If it is an absolute covariant it must be homogeneous of degree zero.*

Again, denoting by C a constant, make the transformation

$$\xi = Cx, \quad \eta = y, \quad \zeta = z,$$

which is also included in the group G This gives

$$\eta^{(\lambda)} = C^{-\lambda} y^{(\lambda)}, \quad \zeta^{(\lambda)} = C^{-\lambda} z^{(\lambda)}, \quad \pi_{ik} = C^{-1} p_{ik}, \quad \varkappa_{ik} = C^{-2} q_{ik},$$
$$\pi^{(\lambda)}_{ik} = C^{-\lambda-1} p^{(\lambda)}_{ik}, \quad \varkappa^{(\mu)}_{ik} = C^{-\mu-2} q^{(\mu)}_{ik}.$$

Let us associate with every quantity an index indicating the power of C^{-1} by which this special transformation multiplies it, and let us speak of this index as its weight Then the weights of $y^{(\lambda)}$ and $z^{(\lambda)}$ are λ, those of p_{ik} and q_{ik} are 1 and 2 respectively, those of $p^{(\lambda)}_{ik}$ and $q^{(\mu)}_{ik}$ are $\lambda + 1$ and $\mu + 2$ respectively. Further, the weight of a product is clearly the sum of the weights of its factors. We see, therefore, that the weights of all of the terms of a covariant must be the same. The covariant must be, as we shall say, *isobaric.* We have obtained the following result.

A covariant must be an isobaric function of the arguments upon which it depends, and of weight zero if it is an absolute covariant.

Let $C_{\lambda, w}$ be an integral, rational, irreducible covariant, homogeneous of degree λ in y, z, y', z' and isobaric of weight w. Let us consider the effect upon $C_{\lambda w}$ of a transformation of the dependent variables alone

If the transformation is

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \tag{68}$$

the corresponding transformation of the coefficients p_{ik} and q_{ik} is given by equations (42) and (43). But in these latter equations we have the new coefficients expressed in terms of the old, while (68) expresses the old variables in terms of the new. We must, therefore, solve (68) which gives

$$\Delta\eta = \delta y - \beta z, \quad \Delta\zeta = -\gamma y + \alpha z, \quad \Delta = \alpha\delta - \beta\gamma \neq 0. \tag{69}$$

Let $\Gamma_{\lambda,w}$ denote what $C_{\lambda,w}$ becomes when y, z, p_{ik}, etc., are replaced by η, ζ, π_{ik}, etc. Since $C_{\lambda,w}$ is a covariant, the equation $\Gamma_{\lambda,w} = 0$ must be a consequence of $C_{\lambda,w} = 0$. But the equations (42), (43), (69) and those deducible from these by differentiation, show clearly that, in place of every term of weight w in C, we shall have in Γ a collection of terms of weight w plus terms of lower weight. But these latter terms must annihilate each other if $C_{\lambda,w}$ is an irreducible covariant, i. e. their sum must be identically zero. For, they cannot vanish as a consequence of $C_{\lambda,w} = 0$, since their aggregate is rational and of lower weight than w, while $C_{\lambda,w}$, being irreducible, cannot be factored into rational factors of lower weight. But it is clear from equations (42), (43) and (69) that the terms of weight w in $\Gamma_{\lambda,w}$, when expressed in terms of y, z, p_{ik}, etc., will contain only the quantities α, β, γ, δ themselves and not their derivatives. There must, therefore, be an equation of the form

$$\Gamma_{\lambda,w} = f(\alpha, \beta, \gamma, \delta)\, C_{\lambda,w},$$

where f contains no other arguments than those indicated

Equations (48), (43) and (69) show further that π_{ik} and $\varkappa_{ik}$ are homogeneous functions of degree zero, and that η and ζ, as well as η' and ζ' are homogeneous functions of degree -1 of the quantities α, β, γ, δ. Therefore $\Gamma_{\lambda,w}$ and consequently $f(\alpha, \beta, \gamma, \delta)$ must be a homogeneous function of its arguments of degree $-\lambda$. Further, the same equations show that $f(\alpha, \beta, \gamma, \delta)$ can be written in the form

$$f(\alpha, \beta, \gamma, \delta) = \frac{\varphi(\alpha, \beta, \gamma, \delta)}{\Delta^{\mu}},$$

where $\varphi(\alpha, \beta, \gamma, \delta)$ is an integral rational function of its arguments, homogeneous of degree $-\lambda + 2\mu$, since the degree of $f(\alpha, \beta, \gamma, \delta)$ is $-\lambda$ and that of Δ is 2.

We have, therefore,

$$\Gamma_{\lambda,w} = \frac{\varphi(\alpha, \beta, \gamma, \delta)}{\Delta^{\mu}} C_{\lambda,w}. \tag{70}$$

But we may regard the system of differential equations in η and ζ as the original system, and that in y and z as the transformed system. We may therefore write equally well

$$(71) \qquad C_{\lambda, w} = \frac{\varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})}{\bar{\Delta}^{\mu}} \Gamma_{\lambda, w},$$

where $\bar{\alpha}$ $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ are the minors of α, β, γ, δ respectively in

$$\Delta = \alpha\delta - \beta\gamma,$$

each divided by Δ, and where

$$\bar{\Delta} = \bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\gamma}.$$

We shall have

$$\bar{\alpha} = \frac{\delta}{\Delta}, \quad \bar{\beta} = \frac{-\gamma}{\Delta}, \quad \bar{\gamma} = \frac{-\beta}{\Delta}, \quad \bar{\delta} = \frac{\alpha}{\Delta}, \quad \bar{\Delta} = \frac{1}{\Delta}.$$

From (70) and (71) we deduce, therefore, by multiplication

$$\varphi(\alpha, \beta, \gamma, \delta)\, \varphi\left(\frac{\delta}{\Delta}, \frac{-\gamma}{\Delta}, \frac{-\beta}{\Delta}, \frac{\alpha}{\Delta}\right) = 1,$$

or since φ is homogeneous of degree $2\mu - \lambda$,

$$\varphi(\alpha, \beta, \gamma, \delta)\, \varphi(\delta, -\gamma, -\beta, \alpha) = \Delta^{2\mu - \lambda},$$

where, be it remembered, φ is an integral rational function of its arguments. But this equation is possible only if $\varphi(\alpha, \beta, \gamma, \delta)$ is a power of Δ, since Δ cannot be factored into two integral rational factors. Since, moreover, $f(\alpha, \beta, \gamma, \delta)$ must be of degree $-\lambda$, it must be equal to $\Delta^{-\frac{\lambda}{2}}$ except for a numerical factor k. But k must be equal to unity, since the identical transformation

$$\eta = y, \quad \zeta = z$$

must give $\Gamma = C$. We have, therefore,

$$\Gamma_{\lambda, w} = \Delta^{-\frac{\lambda}{2}} C_{\lambda, w}.$$

For our proof it was convenient to take the transformation in the form solved for y and z. If we write, instead, the transformation in the form

$$\eta = \alpha y + \beta z, \quad \zeta = \gamma y + \delta z,$$

we know now that a rational covariant $C_{\lambda, w}$, of weight w and of degree λ, is transformed in accordance with the equation

$$\Gamma_{\lambda, w} = (\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}} C_{\lambda, w}.$$

Moreover, as the right member must obviously be rational in α, β, γ, δ, we get this theorem:

There are no rational covariants of odd degree for a binary system of linear homogeneous differential equations.

It is obvious how this theorem will generalize for m-ary systems. Moreover, since we have not made any transformation of the independent variable in proving this theorem, it is also true of all semicovariants which are isobaric.

Let us now make a transformation of the independent variable

$$\xi = \xi(x).$$

The formulae of § 4 show that every term of weight w in $\Gamma_{\lambda,w}$ is equal to a corresponding term of $C_{\lambda,w}$ multiplied by

$$(\xi')^{-w},$$

plus terms of lower weight. But the aggregate of these latter terms must vanish identically, since it cannot vanish in consequence of $C_{\lambda,w} = 0$ which is an irredubible equation, isobaric of weight w. Therefore we shall have

$$\Gamma_{\lambda,w} = \frac{1}{(\xi')^w} C_{\lambda,w}.$$

Combining our results we have the following theorem.

If $C_{\lambda,w}$ is an integral, rational, irreducible covariant of degree λ and of weight w, it is transformed by the transformation

$$\xi = \xi(x), \quad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z,$$

in accordance with the equation

(72)
$$\Gamma_{\lambda,w} = \frac{(\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}}{(\xi')^w} C_{\lambda,w}.$$

Moreover its degree λ is necessarily even.

For invariants $\lambda = 0$. *From two invariants an absolute invariant can always be formed. Similarly, from three covariants an absolute covariant may be obtained*

Finally we may show, just as in Chapter II, that *an absolute rational invariant is the quotient of two relative integral rational invariants of the same weight.*

Let Θ_ν be an integral rational invariant of weight ν. Then, according to (72), the transformations considered will convert it into Θ_ν, where

(73)
$$\Theta_\nu = \frac{1}{(\xi')^\nu} \Theta_\nu,$$

or for infinitesimal transformations into $\Theta_\nu + \delta\Theta_\nu$, where

(74)
$$\delta\Theta_\nu = -\nu\varphi'(x)\Theta_\nu\delta t.$$

We may now proceed to calculate some of the invariants. It is clear that there are no integral rational invariants of weight 1, 2, 3. An invariant of weight 4 must satisfy the equation

$$\delta\Theta_4 = -4\varphi'\Theta_4\delta t.$$

We have already found it [cf. equation (52)], viz.:

(75)
$$\Theta_4 = I^2 - 4J.$$

An invariant of weight 5 must be of the form

$$\Theta_5 = aII' + bI^{(3)} + cJ'$$

and satisfy the equation

$$\delta\Theta_5 = -5\varphi'\Theta_5\delta t.$$

On applying equations (66) and (67) we find $a = b = c = 0$, i. e. there is no such invariant.

An invariant of weight 6 must verify the equation

$$\frac{\delta\Theta_6}{\delta t} = -6\varphi'\Theta_6.$$

The most general expression, integral and rational in the seminvariants, and of weight 6, is

$$\Theta_6 = aI^3 + bIJ + cK + dI^{(4)} + eJ'' + fII'' + gI'^2.$$

We find from (66) and (67);

$$\begin{aligned}\frac{\delta\Theta_6}{\delta t} = {} & 3aI^2(4\varphi^{(3)} - 2\varphi'I) + bI(2\varphi^{(3)}I - 4\varphi'J) + bJ(4\varphi^{(3)} - 2\varphi'I)\\ & + c(8\varphi^{(4)}I' - 8\varphi''J' - 6\varphi'K)\\ & + d(4\varphi^{(7)} - 2\varphi^{(5)}I - 9\varphi^{(4)}I' - 12\varphi^{(3)}I'' - 10\varphi''I^{(3)} - 6\varphi'I^{(4)})\\ & + e(2\varphi^{(5)}I + 4\varphi^{(4)}I' + 2\varphi^{(3)}I'' - 4\varphi^{(3)}J - 9\varphi''J' - 6\varphi'J'')\\ & + fI(4\varphi^{(5)} - 2\varphi^{(3)}I - 5\varphi''I' - 4\varphi'I'') + fI''(4\varphi^{(3)} - 2\varphi'I)\\ & + 2gI'(4\varphi^{(4)} - 2\varphi''I - 3\varphi'I').\end{aligned}$$

This must be equal to $-6\varphi'\Theta_6$ for all values of $\varphi, \varphi', \ldots, I, I', \ldots, J, J', \ldots K$. We find, therefore, the equations:

$$\begin{array}{llll} d = 0, & e + 2f = 0, & 2c + e + 2g = 0, & 6a + b - f = 0,\\ b - e = 0, & & 8c + 9e = 0, & 5f + 4g = 0,\end{array}$$

whence

$$a = -\frac{1}{4}e, \quad b = e, \quad c = -\frac{9}{8}e, \quad d = 0, \quad f = -\frac{1}{2}e, \quad g = \frac{5}{8}e.$$

Putting $e = -8$, we find

$$\Theta_6 = 2I(I^2 - 4J) + 5(K - I'^2) + 4(K - 2J'' + II''). \tag{76}$$

There is no invariant of weight 7, and there are two independent invariants of weight 8, one of which is $\Theta_4^{\,2}$, while the other is

$$\begin{aligned}\Theta_8 = {} & 143(L - 4I''^2) - 54(I^2 + 4J)\Theta_4 - 20I''\Theta_4 + 25I'\Theta_4'\\ & - 206I\Theta_4'' - 20\Theta_4^{(4)} - 902I(K - I'^2) - 220(K'' - 2I'I^{(3)} - 2I''^2).\end{aligned} \tag{77}$$

We may easily find an invariant of weight 10, without going through this general process We have

$$\frac{\delta(K - I'^2)}{\delta t} = 4\varphi''(II' - 2J') - 6\varphi'(K - I'^2),$$

$$\frac{\delta(II' - 2J')}{\delta t} = -2\varphi''(I^2 - 4J) - 5\varphi'(II' - 2J'),$$

whence eliminating φ'', we find

$$\frac{\delta \Theta_{10}}{\delta t} = -10\varphi' \Theta_{10},$$

where

(78) $$\Theta_{10} = (I^2 - 4J)(K - I'^2) + (II' - 2J')^2.$$

From two invariants of weight μ and λ we can, as in Chapter II, equation (24), always form a new invariant of weight $\lambda + \mu + 1$, viz.:

$$\Theta_{\lambda + \mu + 1} = \mu \Theta_\mu \Theta_\lambda' - \lambda \Theta_\lambda \Theta_\mu',$$

the Jacobian of Θ_μ and Θ_λ. We thus obtain the following further invariants:

(79) $$\begin{aligned} \Theta_{11} &= 3\Theta_6\Theta_4' - 2\Theta_4\Theta_6', & \vartheta_{15} &= 4\Theta_8\Theta_6' - 3\Theta_6\Theta_8', \\ \Theta_{13} &= 2\Theta_8\Theta_4' - \Theta_4\Theta_8', & \Theta_{17} &= 5\Theta_{10}\Theta_6' - 3\Theta_6\Theta_{10}', \\ \Theta_{15} &= 5\Theta_{10}\Theta_4' - 2\Theta_4\Theta_{10}', & \Theta_{19} &= 5\Theta_{10}\Theta_8' - 4\Theta_8\Theta_{10}', \end{aligned}$$

from which still others may be derived by a continuation of the process.

We may also, as in Chapter II, deduce from every invariant of weight m another, its quadriderivative, of weight $2m + 2$. But its expression will be slightly different from the expression (54) of that chapter. If we put again for a moment

$$\chi = 2m \frac{d^2 \log \Theta_m}{dx^2} - \left(\frac{d \log \Theta_m}{dx}\right)^2,$$

we shall have, as before,

$$\chi = \frac{1}{(\xi')^2}[\chi + m^2\eta - 2m^2\eta'] = \frac{1}{(\xi')^2}(\chi - 2m^2\mu)$$

We have further, from (51)

$$I = \frac{1}{(\xi')^2}(I + 4\mu),$$

so that

$$2\chi + m^2 I$$

is an invariant. The numerator of this expression

(80) $$\Theta_{m,1} = 2m\Theta_m''\Theta_m - (2m+1)(\Theta_m')^2 + \frac{1}{2}m^2 I \Theta_m^2$$

is the required quadriderivative of Θ_m.

Of all of the invariants found so far

$$\Theta_4, \quad \Theta_6, \quad \Theta_{10}, \quad \Theta_{15}, \quad \Theta_{4\cdot 1}$$

are the only ones which involve no higher derivatives of p_{ik} than the third, and no higher derivatives of q_{ik} than the second. In other words these are the only invariants found so far which depend only upon the seminvariants

(81) $$I, I', I''; \quad J, J', J''; \quad K, K'; \quad L.$$

But only four of these invariants are independent. In fact we find

(82) $$\Theta_{4\cdot 1} + 36\Theta_{10} - 4\Theta_4\Theta_6 = 0.$$

In order to obtain *all* of the absolute invariants depending only upon the seminvariants (81) we write down the system of partial differential equations which they must satisfy.

In order to find these equations we assume that f is any function of the nine arguments (81) and form δf. We shall find

$$\delta f = \sum_{k=1}^{5} Y_k f \varphi^{(k)}(x)\delta t,$$

and the required system of equations is obtained by equating $Y_k f$ to zero for $k = 1, 2, 3, 4, 5$.

We find in this way the following system of equations:

$$
\begin{aligned}
Y_1 f &= 2I\frac{\partial f}{\partial I} + 3I'\frac{\partial f}{\partial I'} + 4I''\frac{\partial f}{\partial I''} + 4J\frac{\partial f}{\partial J} + 5J'\frac{\partial f}{\partial J'} + 6J''\frac{\partial f}{\partial J''} \\
&\quad + 6K\frac{\partial f}{\partial K} + 7K'\frac{\partial f}{\partial K'} + 8L\frac{\partial f}{\partial L} = 0, \\
Y_2 f &= -2I\frac{\partial f}{\partial I'} - 5I'\frac{\partial f}{\partial I''} - 4J\frac{\partial f}{\partial J'} - 9J'\frac{\partial f}{\partial J''} - 8J'\frac{\partial f}{\partial K} \\
&\quad - (8J'' + 6K)\frac{\partial f}{\partial K'} - 20K'\frac{\partial f}{\partial L} = 0, \\
Y_3 f &= 4\frac{\partial f}{\partial I} - 2I\frac{\partial f}{\partial I''} + 2I\frac{\partial f}{\partial J} + 2I'\frac{\partial f}{\partial J'} + 2(I'' - 2J)\frac{\partial f}{\partial J''} \\
&\quad - 8J'\frac{\partial f}{\partial K'} - 16(2J'' - K)\frac{\partial f}{\partial L} = 0, \\
Y_4 f &= 4\frac{\partial f}{\partial I'} + 2I\frac{\partial f}{\partial J'} + 4I'\frac{\partial f}{\partial J''} + 8I'\frac{\partial f}{\partial K} + 8I''\frac{\partial f}{\partial K'} = 0, \\
Y_5 f &= 4\frac{\partial f}{\partial I''} + 2I\frac{\partial f}{\partial J''} + 8I'\frac{\partial f}{\partial K'} + 32I''\frac{\partial f}{\partial L} = 0.
\end{aligned}
\tag{83}
$$

These five equations are independent and, therefore, have $9 - 5 = 4$ independent solutions, i. e. there must be four independent absolute invariants, or five independent relative invariants involving the seminvariants (81). Of these we have found four, viz.: Θ_4, Θ_6, Θ_{10}, Θ_{15}. The fifth invariant may be found by integrating (83). It may be taken to be

$$
\begin{aligned}
\Theta_{18} = \Theta_{10}(L - 4I''^2) &+ 4(K - I'^2)(II'' - 2J'' + K)^2 - \Theta_4(K' - 2I'I'')^2 \\
&- 2\Theta_4'(K' - 2I'I'')(II'' - 2J'' + K).
\end{aligned}
\tag{84}
$$

We may verify directly that the seminvariant $\varDelta$ [equation (41)] is an invariant by means of equations (49), (53) and (54). It clearly depends only upon the seminvariants (81) and must therefore be expressible in terms of Θ_4, Θ_6, Θ_{10}, Θ_{15} and Θ_{18}. Its weight is 9 and we shall henceforth write

$$\varDelta = \Theta_9. \tag{85}$$

We shall find its expression in terms of $\Theta_1, \ldots \Theta_{18}$ in the next paragraph.

§ 6. Canonical forms of a system of two linear differential equations of the second order.

Our system of equations (45) can always be transformed into another which contains no first derivatives, by a transformation of the form

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \quad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ are appropriately chosen functions of x. In fact upon making this transformation, we find

$$(86) \quad \begin{aligned} &\alpha\eta'' + \beta\zeta'' + (2\alpha' + p_{11}\alpha + p_{12}\gamma)\eta' + (2\beta' + p_{11}\beta + p_{12}\delta)\zeta' \\ &\quad + (\alpha'' + p_{11}\alpha' + p_{12}\gamma' + q_{11}\alpha + q_{12}\gamma)\eta \\ &\quad + (\beta'' + p_{11}\beta' + p_{12}\delta' + q_{11}\beta + q_{12}\delta)\zeta = 0, \\ &\gamma\eta'' + \delta\zeta'' + (2\gamma' + p_{21}\alpha + p_{22}\gamma)\eta' + (2\delta' + p_{21}\beta + p_{22}\delta)\zeta' \\ &\quad + (\gamma'' + p_{21}\alpha' + p_{22}\gamma' + q_{21}\alpha + q_{22}\gamma \ \eta \\ &\quad + (\delta'' + p_{21}\beta' + p_{22}\delta' + q_{21}\beta + q_{22}\delta)\zeta = 0. \end{aligned}$$

If, therefore, we take for (α, γ) and (β, δ) two pairs of solutions of the system of equations

$$(87) \quad \begin{aligned} \varrho' &= -\frac{1}{2}(p_{11}\varrho + p_{12}\sigma), \\ \sigma' &= -\frac{1}{2}(p_{21}\varrho + p_{22}\sigma), \end{aligned}$$

the terms containing η' and ζ' in (86) will vanish. Moreover, since $\alpha\delta - \beta\gamma$ must not be zero, (α, γ) and (β, δ) must be two independent systems of solutions of (87), and two such systems always exist.

If one makes use of (87) and the equations obtained from (87) by differentiation, (86) becomes

$$(88) \quad \begin{aligned} \alpha\eta'' + \beta\zeta'' &= \frac{1}{4}(\alpha u_{11} + \gamma u_{12})\eta + \frac{1}{4}(\beta u_{11} + \delta u_{12})\zeta, \\ \gamma\eta'' + \delta\zeta'' &= \frac{1}{4}(\alpha u_{21} + \gamma u_{22})\eta + \frac{1}{4}(\beta u_{21} + \delta u_{22})\zeta, \end{aligned}$$

where the quantities u_{ik} are the same as those which have been previously denoted in this way [cf. equations (20)].

Thus every binary system of homogeneous linear differential equations of the second order may be converted into another, involving no first derivatives, i. e. into one for which $p_{ik} = 0$. We shall say that it has been reduced to the *semi-canonical form*.

Suppose that the system is given in its semi-canonical form

$$\begin{aligned} y'' + q_{11}y + q_{12}z &= 0, \\ z'' + q_{21}y + q_{22}z &= 0. \end{aligned}$$

The transformation

$$\xi = \xi(x), \quad y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

converts it into

$$(89)\quad \begin{aligned} &\alpha(\xi')^2\frac{d^2\eta}{d\xi^2}+\beta(\xi')^2\frac{d^2\zeta}{d\xi^2}+(\alpha\xi''+2\alpha'\xi')\frac{d\eta}{d\xi}+(\beta\xi''+2\beta'\xi')\frac{d\zeta}{d\xi}\\ &\qquad+(\alpha''+q_{11}\alpha+q_{12}\gamma)\eta+(\beta''+q_{11}\beta+q_{12}\delta)\zeta=0,\\ &\gamma(\xi')^2\frac{d^2\eta}{d\xi^2}+\delta(\xi')^2\frac{d^2\zeta}{d\xi^2}+(\gamma\xi''+2\gamma'\xi')\frac{d\eta}{d\xi}+(\delta\xi''+2\delta'\xi')\frac{d\zeta}{d\xi}\\ &\qquad+(\gamma''+q_{21}\alpha+q_{22}\gamma)\eta+(\delta''+q_{21}\beta+q_{22}\delta)\zeta=0. \end{aligned}$$

This is again in the semi-canonical form if

$$\alpha\xi''+2\alpha'\xi'=\beta\xi''+2\beta'\xi'=\gamma\xi''+2\gamma'\xi'=\delta\xi''+2\delta'\xi'=0,$$

i. e. if

$$\alpha=\frac{a}{\sqrt{\xi'}},\quad \beta=\frac{b}{\sqrt{\xi'}},\quad \gamma=\frac{c}{\sqrt{\xi'}},\quad \delta=\frac{d}{\sqrt{\xi'}},$$

where a, b, c, d are constants, whose determinant $ad-bc$ does not vanish. We see, therefore, that *the equations*

$$(90)\qquad \xi=\xi(x),\quad y=(a\eta+b\zeta)\sqrt{\xi'},\quad z=(c\eta+d\zeta)\sqrt{\xi'}$$

give the most general transformations which leave the semi-canonical form invariant, $\xi(x)$ being an arbitrary function and a, b, c, d arbitrary constants.

Let us put in particular

$$a=d=1,\quad b=c=0,$$

or

$$\alpha=\delta=\frac{1}{\sqrt{\xi'}},\quad \beta=\gamma=0.$$

Then (89) becomes

$$\alpha(\xi')^2\frac{d^2\eta}{d\xi^2}+(\alpha''+q_{11}\alpha)\eta+q_{12}\alpha\zeta=0,$$

$$\alpha(\xi')^2\frac{d^2\zeta}{d\xi^2}+q_{21}\alpha\eta+(\alpha''+q_{22}\alpha)\zeta=0,$$

or

$$\frac{d^2\eta}{d\xi^2}+\varrho_{11}\eta+\varrho_{12}\zeta=0,\quad \frac{d^2\zeta}{d\xi^2}+\varrho_{21}\eta+\varrho_{22}\zeta=0$$

Now α can be determined in such a way as to make

$$\varrho_{11}+\varrho_{22}=0.$$

For this purpose it is only necessary to take for α a solution of the linear differential equation

$$(91)\qquad 2\alpha''+(q_{11}+q_{22})\alpha=0.$$

If we put again

$$\frac{\xi''}{\xi'}=\eta,$$

we find for η the *Riccati* equation

$$(92)\qquad \mu=\eta'-\frac{1}{2}\eta^2=q_{11}+q_{22},$$

whose left member in terms of ξ is the *Schwarzian derivative* of ξ with respect to x [cf. equation (50)].

We have proved the following theorem:

Every system of two linear homogeneous differential equations of the second order can be transformed into a system of the form

$$\frac{d^2\eta_i}{d\xi^2} + \varrho_{i1}\eta_1 + \varrho_{i2}\eta_2 = 0 \quad (i = 1,\ 2),$$

where

$$\varrho_{11} + \varrho_{22} = 0.$$

In order to effect this reduction, it is necessary to integrate a system of two homogeneous linear differential equations of the first order (87), *an equation of the Riccati type* (92) *and finally to effect the quadrature*

$$\xi(x) = \int \frac{dx}{\alpha^2}.$$

This canonical form of the system corresponds to the *Laguere-Forsyth* canonical form of a single linear differential equation.

If $\xi(x)$ is any solution of (92), its most general solution is $\frac{\alpha\xi + \beta}{\gamma\xi + \delta}$. We see, *therefore, that the most general transformations, which leave the canonical form invariant, are*

$$(93) \qquad \xi = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \eta = \frac{ay + bz}{\gamma x + \delta}, \quad \zeta = \frac{cy + dz}{\gamma x + \delta},$$

where α, β, γ, δ, a, b, c, d are arbitrary constants. These transformations form a seven-parameter group.

There is another canonical form which is of special importance. Equations (44) show that the coefficients α, β, γ, δ of the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\xi$$

may be chosen so as to make $u_{12} = \bar{u}_{21} = 0$. It suffices to determine the ratios $\beta:\delta$ and $\alpha:\gamma$ as the two roots of the quadratic equation

$$- u_{21}\lambda^2 + (u_{11} - u_{22})\,\lambda + u_{12} = 0.$$

Since $\alpha\delta - \beta\gamma$ must not be zero, the roots of this quadratic must be distinct, i. e.

$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = \Theta_4 \neq 0.$$

By merely solving a quadratic equation we may therefore reduce our system to another for which $u_{12} = 0$ and $u_{21} = 0$, provided that Θ_4 is different from zero.

Suppose that the system has been reduced to this form. A transformation of the form

$$y = \alpha\eta, \quad z = \delta\zeta, \quad \xi = \xi(x)$$

according to (44) and (49) will not disturb the conditions $u_{12} = u_{21} = 0$, the functions α, δ and ξ being arbitrary. According to (86) the transformed system will be

$$\alpha\left[\frac{d^2\eta}{d\xi^2}(\xi')^2+\frac{d\eta}{d\xi}\xi''\right]+(2\alpha'+p_{11}\alpha)\frac{d\eta}{d\xi}\xi'+p_{12}\delta\frac{d\zeta}{d\xi}\xi'$$
$$+(\alpha''+p_{11}\alpha'+q_{11}\alpha)\eta+(p_{12}\delta'+q_{12}\delta)\zeta=0,$$
$$\delta\left[\frac{d^2\zeta}{d\xi^2}(\xi')^2+\frac{d\zeta}{d\xi}\xi''\right]+p_{21}\alpha\frac{d\eta}{d\xi}\xi'+(2\delta'+p_{22}\delta)\frac{d\zeta}{d\xi}\xi'$$
$$+(p_{21}\alpha'+q_{21}\alpha)\eta+(\delta''+p_{22}\delta'+q_{22}\delta)\zeta=0.$$

If we choose α and δ so that

$$2\alpha'+p_{11}\alpha+\alpha\frac{\xi''}{\xi'}=0,$$
$$2\delta'+p_{22}\delta+\delta\frac{\xi''}{\xi'}=0,$$

the coefficient of $\frac{d\eta}{d\xi}$ in the first, and of $\frac{d\zeta}{d\xi}$ in the second equation will be zero. We may therefore reduce our system, for which $u_{12}=u_{21}=0$, further to a system for which also $p_{11}=p_{22}=0$, by the substitution

$$y=\frac{a}{\sqrt{\xi'}}e^{-\frac{1}{2}\int p_{11}dx}\eta,\quad z=\frac{b}{\sqrt{\xi'}}e^{-\frac{1}{2}\int p_{22}dx}\zeta,\quad \xi=\xi(x),$$

where $\xi(x)$ is an arbitrary function, and where a and b are arbitrary constants.

The arbitrary function $\xi(x)$, finally, may be used to reduce the invariant Θ_4 to unity. In fact, we have

$$\overline{\Theta}_4=\frac{1}{(\xi')^4}\Theta_4.$$

We shall, therefore, find $\overline{\Theta}_4=1$, if we put

$$\xi=\int\sqrt[4]{\Theta_4}\,dx+\text{const.}$$

Since, moreover, in this case $u_{12}=u_{21}=0$, $\Theta_4=(u_{11}-u_{22})^2$, we may more specifically reduce $u_{11}-u_{22}$ to unity, by putting

$$\xi=\int\sqrt{u_{11}-u_{22}}\,dx+\text{const.}$$

We may also, if we prefer, reduce any of the other invariants Θ_6, Θ_{10} etc. to unity. Or, we may reduce the seminvariants I or J to zero by making use of equations (51).

Let us assume that the system of differential equations has been written so as to make $u_{12}=u_{21}=0$, $p_{11}=p_{22}=0$, $u_{11}-u_{22}=1$. We shall make use of this special form to express $\Delta=\Theta_9$ in terms of the seminvariants I, J, K, L. We find from (41),

$$\Delta=v_{12}w_{21}-v_{21}w_{12}.$$

But we shall have, in this case,

$$v_{11} = 2u'_{11}, \quad v_{12} = -p_{12}, \quad v_{21} = +p_{21}, \quad v_{22} = 2u'_{22},$$
$$w_{11} = 4u''_{11} + 2p_{12}p_{21}, \quad w_{12} = -2p'_{12}, \quad w_{21} = +2p'_{21}, \quad w_{22} = 4u''_{22} - 2p_{12}p_{21},$$

whence

$$\varDelta = 2(p_{21}p'_{12} - p_{12}p'_{21}).$$

Further we have

$$I = u_{11} + u_{22}, \quad J = u_{11}u_{22}, \quad K = v_{11}v_{22} + p_{12}p_{21},$$
$$L = 16u''_{11}u''_{22} - 4p^2_{12}p^2_{21} + 4p'_{12}p'_{21},$$
$$\Theta_4 = 1, \quad K - I'^2 = p_{12}p_{21}, \quad \Theta_{10} = p_{12}p_{21},$$
$$L - 4I''^2 = -4p^2_{12}p^2_{21} + 4p'_{12}p'_{21},$$

whence, according to (84)

$$\Theta_{18} = -(p_{12}p'_{21} - p_{21}p'_{12})^2,$$

so that

$$\varDelta^2 + 4\Theta_{18} = 0. \tag{94}$$

This equation must hold between $\varDelta$ and Θ_{18} whatever may be the form in which the differential equations are written. For, a relation between invariants is not changed by any transformation of the kind which we have made. It is true however that in such a relation Θ_4 would seem to disappear, since this invariant has been made equal to unity. There might, therefore, be a factor of the form $\Theta_4{}^\lambda$ in one of the terms of the above equation. But this is impossible in the present case because $\varDelta^2$ and Θ_{18} are both of weight 18. Such a factor would disturb the condition that the left member of the equation be isobaric.

§ 7 The complete system of invariants.

Suppose that the invariants Θ_4, Θ_9, $\Theta_{4\cdot 1}$ and Θ_{10} are given as functions of x We shall show that the corresponding system of differential equations is determined by these four functions, so far as it is possible to determine such a system by means of invariants. For, it is clear that, one such system being given, any other, which can be obtained from it by a transformation of the form

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

will have the same invariants.

Let

$$\begin{aligned} &y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ &z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0 \end{aligned} \tag{95}$$

be such a system of equations, whose invariants Θ_4, Θ_9, $\Theta_{4\cdot 1}$ and Θ_{10} are arbitrarily prescribed functions of x, and let us assume that Θ_4 is not identically zero. We may then, as we have seen in § 6,

transform it into another system of the same form, for which $u_{12} = u_{21} = p_{11} = p_{22} = 0$, and we shall assume that this transformation has already been made, so that we shall have

(96) $$u_{12} = u_{21} = p_{11} = p_{22} = 0.$$

We find consequently;

(97) $$\begin{gathered} v_{11} = 2u'_{11}, \quad v_{12} = -p_{12}(u_{11} - u_{22}), \quad v_{21} = +p_{21}(u_{11} - u_{22}), \quad v_{22} = 2u'_{22}, \\ w_{11} = 2v'_{11} + 2p_{12}p_{21}(u_{11} - u_{12}), \quad w_{12} = 2v'_{12} - 2p_{12}(u'_{11} - u'_{22}), \\ w_{21} = 2v'_{21} + 2p_{21}(u'_{11} - u'_{22}), \quad w_{22} = 2v'_{22} - 2p_{12}p_{21}(u_{11} - u_{22}), \end{gathered}$$

whence

(98) $$\begin{aligned} \Theta_4 &= (u_{11} - u_{22})^2, \\ \Theta_{4\cdot 1} &= 8\Theta_4\Theta_4'' - 9(\Theta_4')^2 + 8(u_{11} + u_{22})\Theta_4^2, \\ \Theta_{10} &= -(u_{11} - u_{12})^2 v_{12}v_{21} = (u_{11} - u_{22})^4 p_{12}p_{21}, \\ \Theta_9 &= 2(u_{11} - u_{22})^3(p'_{12}p_{21} - p'_{21}p_{12}). \end{aligned}$$

From these equations we find

(99) $$\begin{gathered} u_{11} - u_{22} = \varepsilon\sqrt{\Theta_4}, \quad \varepsilon = \pm 1, \\ u_{11} + u_{22} = \frac{1}{8\Theta_4^2}[\Theta_{4\cdot 1} - 8\Theta_4\Theta_4'' + 9(\Theta_4')^2], \\ p_{12}p_{21} = \frac{\Theta_{10}}{\Theta_4^2}, \quad p'_{12}p_{21} - p'_{21}p_{12} = \frac{\Theta_9}{2\varepsilon\Theta_4^{\frac{3}{2}}}. \end{gathered}$$

The last two equations give

$$\frac{p'_{12}}{p_{12}} - \frac{p'_{21}}{p_{21}} = \frac{1}{2}\varepsilon\sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}},$$

whence by integration

(100) $$\frac{p_{12}}{p_{21}} = C^2 e^{\frac{1}{2}\varepsilon\int\sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}}dx},$$

where C is an arbitrary constant.

We find further, from (99) and (100),

$$p_{12}^2 = C^2\frac{\Theta_{10}}{\Theta_4^2}e^{\frac{1}{2}\varepsilon\int\sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}}dx},$$

whence

$$p_{12} = \varepsilon' C\frac{\sqrt{\Theta_{10}}}{\Theta_4}e^{\frac{1}{4}\varepsilon\int\sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}}dx}, \quad p_{21} = \varepsilon'\frac{1}{C}\frac{\sqrt{\Theta_{10}}}{\Theta_4}e^{-\frac{1}{4}\varepsilon\int\sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}}dx},$$

where $\varepsilon' = \pm 1$. Since $u_{12} = u_{21} = p_{11} = p_{22} = 0$, we have further

$$q_{12} = \frac{1}{2}p'_{12}, \quad q_{21} = \frac{1}{2}p'_{21}$$

and

$$u_{11} = -4q_{11} + p_{12}p_{21}, \quad u_{22} = -4q_{22} + p_{12}p_{21},$$

so that we may also compute q_{11} and q_{22} from (99).

We find, therefore, for the coefficients of (95) the following expressions

$$
(101)\quad
\begin{aligned}
&p_{11} = p_{22} = 0,\\
&p_{12} = \varepsilon' C \frac{\sqrt{\Theta_{10}}}{\Theta_4} e^{\frac{1}{4}\varepsilon \int \sqrt{\Theta_4} \frac{\Theta_9}{\Theta_{10}} dx}, \quad p_{21} = \varepsilon' \frac{1}{C} \frac{\sqrt{\Theta_{10}}}{\Theta_4} e^{-\frac{1}{4}\varepsilon \int \sqrt{\Theta_4} \frac{\Theta_9}{\Theta_{10}} dx},\\
&64 q_{11} = \frac{1}{\Theta_4^2}[16\Theta_{10} - \Theta_{4\cdot 1} + 8\Theta_4\Theta_4'' - 9(\Theta_4')^2] - 8\varepsilon\sqrt{\Theta_4}, \quad q_{12} = \frac{1}{2}p_{12}',\\
&64 q_{22} = \frac{1}{\Theta_4^2}[16\Theta_{10} - \Theta_{4\cdot 1} + 8\Theta_4\Theta_4'' - 9(\Theta_4')^2] + 8\varepsilon\sqrt{\Theta_4}, \quad q_{21} = \frac{1}{2}p_{21}',
\end{aligned}
$$

which contain one arbitrary constant C and two ambiguous signs ε and ε'. But we may get rid of these. In the first place let us transform this system by putting

$$y = k\eta, \quad z = l\zeta,$$

where k and l are constants. The coefficients p_{11}, q_{11}, p_{22}, q_{22} of the new system will be the same as those of the old, while p_{12}, q_{12} and p_{21}, q_{21} will be multiplied by $\frac{l}{k}$ and $\frac{k}{l}$ respectively. If, therefore, we put

$$\varepsilon' \frac{l}{k} C = 1,$$

the coefficients of the new system will be given by (101) with $\varepsilon' = 1$ and $C = 1$. There still remain in (101), the two possibilities: $\varepsilon = \pm 1$. Denote the values of the quantities (101) for $\varepsilon = +1$ by p_{ik}, q_{ik}, and those for $\varepsilon = -1$ by $\bar{p}_{ik}$, $\bar{q}_{ik}$. Then

$$
\begin{aligned}
&\bar{p}_{11} = p_{11} = 0, \quad \bar{p}_{22} = p_{22} = 0, \quad \bar{p}_{12} = p_{21}, \quad \bar{p}_{21} = p_{12},\\
&\bar{q}_{11} = q_{22}, \quad \bar{q}_{22} = q_{11}, \quad \bar{q}_{12} = q_{21}, \quad \bar{q}_{21} = q_{12}.
\end{aligned}
$$

But it is evident that two systems of differential equations, whose coefficients are connected in this way, may be transformed into each other by putting

$$\bar{y} = z, \quad \bar{z} = y.$$

Equations (101) are valid only if $\Theta_{10} \neq 0$. The following theorem is therefore true.

If the invariants Θ_4, $\Theta_{4\cdot 1}$, Θ_9 and Θ_{10} are given as arbitrary functions of x, Θ_4 and Θ_{10} however not being equal to zero, the system of differential equations whose coefficients are

$$
(102)\quad
\begin{aligned}
&p_{11} = 0, \quad p_{22} = 0, \quad q_{12} = \frac{1}{2}p_{12}', \quad q_{21} = \frac{1}{2}p_{21}',\\
&p_{12} = \frac{\sqrt{\Theta_{10}}}{\Theta_4} e^{\frac{1}{4}\int \sqrt{\Theta_4} \frac{\Theta_9}{\Theta_{10}} dx}, \quad p_{21} = \frac{\sqrt{\Theta_{10}}}{\Theta_4} e^{-\frac{1}{4}\int \sqrt{\Theta_4} \frac{\Theta_9}{\Theta_{10}} dx},\\
&64 q_{11} = \frac{1}{\Theta_4^2}[16\Theta_{10} - \Theta_{4\cdot 1} + 8\Theta_4\Theta_4'' - 9(\Theta_4')^2] - 8\sqrt{\Theta_4},\\
&64 q_{22} = \frac{1}{\Theta_4^2}[16\Theta_{10} - \Theta_{4\cdot 1} + 8\Theta_4\Theta_4'' - 9(\Theta_4')^2] + 8\sqrt{\Theta_4},
\end{aligned}
$$

is one, whose invariants Θ_4, $\Theta_{4\cdot 1}$, Θ_9 *and* Θ_{10} *coincide with these arbitrarily given functions of* x. *Moreover, the most general system of differential equations, which has the same property, can be obtained, from this uniquely determined special one, by a transformation of the form*

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \quad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ *are arbitrary functions of* x.

We shall have occasion, later, to complete this theorem, which is of fundamental importance. For the present we shall merely make use of it to prove that *the invariants* Θ_4, $\Theta_{4\cdot 1}$, Θ_9, Θ_{10}, *together with those formed from them by repetitions of the Jacobian process, constitute a functionally complete set of invariants.* Let I be any invariant. Then the above theorem shows that I can be expressed in terms of Θ_4, $\Theta_{4\cdot 1}$, Θ_9, Θ_{10} and of the derivatives of these quantities. For, the system of differential equations can be transformed into one whose coefficients are given by (102), and the invariant I may be computed in terms of these We have

$$\delta\Theta_\nu = -\nu\varphi'\Theta_\nu\delta t,$$
$$\delta\Theta_\nu' = -(\nu+1)\varphi'\Theta_\nu'\delta t - \nu\varphi''\Theta_\nu\delta t,$$

as the infinitesimal transformations of Θ_ν and Θ_ν'. An absolute invariant depending upon $\Theta_4 \ldots \Theta_{10}$, $\Theta_4' \ldots \Theta_{10}'$ must therefore satisfy the equations:

$$
\begin{aligned}
& 4\Theta_4\frac{\partial f}{\partial\Theta_4} + 9\Theta_9\frac{\partial f}{\partial\Theta_9} + 10\Theta_{4\cdot 1}\frac{\partial f}{\partial\Theta_{4\cdot 1}} + 10\Theta_{10}\frac{\partial f}{\partial\Theta_{10}} + 5\Theta_4'\frac{\partial f}{\partial\Theta_4'} \\
(103) \qquad & \quad + 10\Theta_9'\frac{\partial f}{\partial\Theta_9'} + 11\Theta_{4\cdot 1}'\frac{\partial f}{\partial\Theta_{4\cdot 1}'} + 11\Theta_{10}'\frac{\partial f}{\partial\Theta_{10}'} = 0, \\
& 4\Theta_4\frac{\partial f}{\partial\Theta_4'} + 9\Theta_9\frac{\partial f}{\partial\Theta_9'} + 10\Theta_{4\cdot 1}\frac{\partial f}{\partial\Theta_{4\cdot 1}'} + 10\Theta_{10}\frac{\partial f}{\partial\Theta_{10}'} = 0.
\end{aligned}
$$

There must be $8 - 2 = 6$ such absolute invariants. Now, we know that Θ_4, $\Theta_{4\cdot 1}$, Θ_9, Θ_{10} and the three *Jacobians* of Θ_4 with the other three are seven independent relative invariants, which give rise to six absolute invariants. Therefore, in this case, the *Jacobian* process gives all of the invariants. If the invariant contains also the second derivatives of $\Theta_4, \ldots \Theta_{10}$, we shall have to integrate a complete system containing one more equation and four more independent variables than (103). There will, consequently, be three more independent invariants (absolute or relative) But these can be obtained from the former three by combining them once more with Θ_4 by means of the *Jacobian* process. If we continue in this way, we see that the number of invariants, containing the derivatives of $\Theta_4, \ldots \Theta_{10}$ up to the n^{th} order, is precisely greater by three than the number of invariants containing the derivatives up to the $n-1^{th}$ order. The

three new invariants are obtained by combining Θ_4 with Θ_9, $\Theta_{4\cdot 1}$, Θ_{10} n times by means of the *Jacobian* process.

We leave open the question whether all rational invariants can be expressed as *rational* functions of these fundamental invariants. We shall indicate in a different connection, however, how a system of invariants complete in this higher sense may be constructed.

§ 8. The covariants.

We have seen in § 6, that the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta$$

may be chosen in such a way as to make $\bar{u}_{12}$ and u_{21} vanish for the transformed system of differential equations. For this purpose it was necessary and sufficient to take $\frac{\beta}{\delta}$ and $\frac{\alpha}{\gamma}$ as the two roots of the quadratic

(104) $$-u_{21}\lambda^2 + (u_{11} - u_{22})\lambda + u_{12} = 0.$$

We have, on the other hand,

$$\Delta\eta = \delta y - \beta z, \quad \Delta\zeta = -\gamma y + \alpha z, \quad \Delta = \alpha\delta - \beta\gamma,$$

so that

(105) $$\Delta^2\eta\zeta = (\delta y - \beta z)(-\gamma y + \alpha z)$$

is an expression whose linear factors are uniquely determined by the conditions that $\bar{u}_{12}$ and $\bar{u}_{21}$ shall vanish Except for a factor, independent of y and z, this expression must therefore be a semi-covariant. We proceed to calculate it We find from (104)

$$\frac{\beta}{\delta} = \frac{u_{11} - u_{22} + \sqrt{\Theta_4}}{2u_{21}}, \quad \frac{\alpha}{\gamma} = \frac{u_{11} - u_{22} - \sqrt{\Theta_4}}{2u_{21}},$$

whence

$$\Delta^2\eta\zeta = -\gamma\delta\left[y^2 - \frac{u_{11} - u_{22}}{u_{21}}yz - \frac{u_{12}}{u_{21}}z^2\right].$$

We may verify directly that

(106) $$C = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz$$

is a semi-covariant, in accordance with our prevision.

Since, for transformations of the dependent variables alone, v_{ik} and w_{ik} are cogredient with u_{ik}, the following expressions

(107) $$\begin{aligned} E &= v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz, \\ F &= w_{12}z^2 - w_{21}y^2 + (w_{11} - w_{22})yz, \end{aligned}$$

will be semi-covariants. The weights of C, E, F are 2, 3 und 4 respectively.

If then we make the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \quad \Delta = \alpha\delta - \beta\gamma,$$

and denote the transformed functions by dashes, we shall have

$$C = \Delta C, \quad E = \Delta E, \quad F = \Delta F,$$

so that the determinants

$$(C'E) = C'E - CE' \quad (E'F) = E'F - EF', \tag{108}$$
$$(F'C) = F'C - FC'$$

are also semi-covariants, obviously of degree four.

We have further

$$\frac{C'}{C} = \frac{\bar{C}'}{C} + \frac{\Delta'}{\Delta},$$

and from (42)

$$p_{11} + p_{22} = p_{11} + \bar{p}_{22} - 2\frac{\Delta'}{\Delta},$$

whence

$$2\frac{C'}{C} + p_{11} + p_{22} = 2\frac{C'}{C} + p_{11} + \bar{p}_{22},$$

so that

$$G = 2C' + (p_{11} + p_{22})\, C \tag{109}$$

is a new semi-covariant Similarly we find *two other semi-covariants*

$$H = 2E' + (p_{11} + p_{22})E, \quad M = 2F' + (p_{11} + p_{22})F. \tag{110}$$

We have noticed already that we need to consider only those semi-covariants and covariants which involve no higher derivatives of y and z than the first. For, if such a function contains higher derivatives, we may express them in terms of y, z, y', z' by means of the fundamental differential equations and those deduced therefrom by differentiation.

So as to proceed in an orderly manner, let us first determine all independent semi-covariants containing besides y, z, y', z', merely the quantities p_{ik}, p'_{ik} and q_{ik} We have already found one such, namely C. We can find another by forming $G - E$. If we make use of the equations (32) for v_{ik}, we shall find

$$\begin{aligned} N = G - E = &\{2p_{22}u_{12} + p_{12}(u_{11} - u_{22})\}z^2 - \{2p_{11}u_{21} - p_{21}(u_{11} - u_{22})\}y^2 \\ &+ \{2p_{21}u_{12} - 2p_{12}u_{21} + (p_{11} + p_{22})(u_{11} - u_{22})\}yz \\ &+ 4u_{12}zz' - 4u_{21}yy' + 2(u_{11} - u_{22})(yz' + y'z), \end{aligned} \tag{111}$$

a semi-covariant of degree 2 and of weight 3 involving only the variables mentioned.

The system of partial differential equations, whose solutions are the seminvariants and absolute semi-covariants involving these variables y, z, y', z' p_{ik}, p'_{ik}, q_{ik}, is obtained from (18) by adding to the left members the terms depending upon the partial derivatives of f with respect to y, z, y', z'. These twelve equations are seen to be independent when these additional terms are written down, although without these terms only ten of them are independent. There are

sixteen independent variables, and consequently four solutions, i. e. four seminvariants and absolute semi-covariants. Of these we know the seminvariants I and J. The other two solutions must be absolute semi-covariants. We have found two relative semi-covariants, C and N. There must be just one more, which might be found by integrating the system of partial differential equations just mentioned. It is more instructive to proceed as follows. Put

$$(112)\qquad \begin{aligned} \varrho &= 2y' + p_{11}y + p_{12}z, \\ \sigma &= 2z' + p_{21}y + p_{22}z. \end{aligned}$$

Then it may be verified, either by infinitesimal, or by finite transformations, that *ϱ and σ are cogredient with y and z.* In other words if y and z are transformed by the equations

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z}, \quad \alpha\delta - \beta\gamma \neq 0,$$

ϱ and σ will be transformed by the same equations

$$\varrho = \alpha\bar{\varrho} + \beta\bar{\sigma}, \quad \sigma = \gamma\varrho + \delta\sigma.$$

Therefore,

$$(113)\quad P = z\varrho - y\sigma = 2(y'z - yz') + p_{12}z^2 - p_{21}y^2 + (p_{11} - p_{22})yz,$$

is a semi-covariant of degree 2 and of weight 1. Moreover it is clear that I, J, C, P and N are independent of each other, so that all semi-covariants involving only y, z, y', z', p_{ik}, p'_{ik}, q_{ik} have been found. Equations (112) enable us to write N more simply. We find

$$(114)\qquad N = 2(u_{12}z\sigma - u_{21}y\varrho) + (u_{11} - u_{22})(z\varrho + y\sigma).$$

In order to find the seminvariants and semi-covariants involving p''_{ik} and q'_{ik} besides the former variables, we may set up the system of partial differential equations satisfied by them. It is the system (25) with the terms in y, z, y', z' added. This system contains 24 independent variables and 16 independent equations. Therefore, there must be eight such seminvariants and absolute semi-covariants. But we know them already. They are

$$I,\ I',\ J,\ J',\ K;\ \frac{P}{C},\ \frac{N}{C},\ \frac{E}{C};$$

for these are independent, as may be verified without any difficulty.

In the same way we notice that there must be 12 seminvariants and absolute semi-covariants involving the further variables $p^{(3)}_{ik}$ and q''_{ik}. They are the above with the addition of I'', J'', K' and L. $\frac{F}{C}$, which obviously also belongs to this same class of semi-covariants, must therefore be expressible in terms of these twelve quantities.

No new semi-covariants will appear if we continue our search, and all of the new seminvariants are, as we know, derivatives of I, J, K, L.

Put

$$\frac{C}{P} = \gamma, \quad \frac{E}{P} = \varepsilon, \quad \frac{N}{P} = \nu,$$

so that γ, ε and ν are three independent absolute semi-covariants. Then any absolute covariant must be a function of the seminvariants I, J, K, L, of their derivatives and of γ, ε, ν. The system of partial differential equations which they satisfy will be the same as the system satisfied by the invariants, except for additional terms involving derivatives with respect to the three further variables γ, ε, ν. This new system, containing as many independent equations as the old, but three more variables will have three more solutions. There are, therefore, three independent absolute, or four independent relative covariants. These may be found without trouble We verify easily that C is one We find further, on making the transformation

$$\xi = \xi(r),$$

that ϱ and σ are converted in $\bar{\varrho}$ and $\bar{\sigma}$, where

$$\bar{\varrho} = \frac{1}{\xi'}(\varrho + \eta y), \quad \bar{\sigma} = \frac{1}{\xi'}(\sigma + \eta z), \quad \eta = \frac{\xi''}{\xi'}, \tag{115}$$

whence we see at once that the semi-covariant P is also a covariant We find further

$$E = \frac{1}{(\xi')^3}[E - 4\eta C],$$

$$N = \frac{1}{(\xi')^3}[N + 2\eta C],$$

so that $E + 2N$ is a further covariant. Finally we have

$$C = \frac{1}{(\xi')^2} C, \quad \Theta_4 = \frac{1}{(\xi')^4}\Theta_1,$$

whence

$$\Theta_1' = \frac{1}{(\xi')^5}[\Theta_1' - 4\eta\Theta_1],$$

so that

$$\Theta_4 E - \Theta_1' C$$

is a covariant These four are clearly independent. *We have found the following four covariants*

$$C_1 = P, \quad C_2 = C, \quad C_3 = E + 2N, \quad C_7 = \Theta_4 E - \Theta_4' C, \tag{116}$$

all of which are quadratic, and where the index indicates the weight. All others can be expressed in terms of these and of invariants.

As the three fundamental absolute covariants we may take

$$\frac{C_2^4}{\Theta_4 C_1^4}, \quad \frac{C_3^2}{\Theta_4 C_1^2}, \quad \frac{C_7}{\Theta_4 C_3}.$$

Examples.

Ex. 1. Show that the system of differential equations

$$y' = p_{11}y + p_{12}z, \quad z' = p_{21}y + p_{22}z$$

has no invariants under the transformations

$$\xi = f(x), \quad \bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z,$$

where α, β, γ, δ, f are arbitrary functions of x.

Ex. 2. Find the invariants and covariants of the above system for the transformations

$$\bar{y} = \alpha y, \quad \bar{z} = \beta z, \quad \xi = f(x),$$

where α, β, f are arbitrary functions of x.

Ex. 3. Find the relations between the invariants (79).

Ex. 4. Show that, if $\Delta = 0$, and if the other invariants are constants, the system may be reduced to one with constant coefficients.

Ex. 5. Find the relation between

$$I, I', I''; \; J, J', J''; \; K, K'; \; L; \; P, C, N, E, F;$$

making use of the canonical form for which $u_{12} = u_{21} = 0$.

CHAPTER V.

FOUNDATIONS OF THE THEORY OF RULED SURFACES.

§ 1. Definition of the general solutions, and of a fundamental system of solutions of a simultaneous system of two linear homogeneous differential equations of the second order.

For the sake of brevity we shall speak, hereafter, of the system of differential equations

$$\text{(A)} \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0 \end{aligned}$$

as system (A).

According to the fundamental theorem of the theory of differential equations, the system (A) defines two functions y and z of x, which are analytic in the vicinity of $x = x_0$ if the coefficients are analytic in that vicinity, and which can be made to satisfy the further conditions that y, z, y' and z' shall assume arbitrarily prescribed values for $x = x_0$.

Such a system of solutions, corresponding to these four arbitrarily prescribed values of y, z, y', z' for $x = x_0$, is said to constitute a system of *general* solutions of system (A).

Now let (y_i, z_i) for $(i = 1, 2, 3, 4)$, be any four systems of solutions of (A), so that

$$
(1) \qquad \begin{aligned} y_i'' + p_{11} y_i' + p_{12} z_i' + q_{11} y_i + q_{12} z_i = 0, \\ z_i'' + p_{21} y_i' + p_{22} z_i' + q_{21} y_i + q_{22} z_i = 0, \end{aligned} \qquad (i = 1, 2, 3, 4).
$$

Then, denoting by c_1, c_2, c_3, c_4 four arbitrary constants,

$$
(2) \qquad y = \sum_{i=1}^{4} c_i y_i, \qquad z = \sum_{i=1}^{4} c_i z_i
$$

will also form a simultaneous system of solutions. Moreover from (2) and

$$
(3) \qquad y' = \sum_{i=1}^{4} c_i y_i', \qquad z' = \sum_{i=1}^{4} c_i z_i',
$$

the constants $c_1, \dots c_4$ can be determined in such a way as to give arbitrary constant values to y, z, y', z', for $x = x_0$, provided that the determinant

$$
(4) \qquad D = \begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}
$$

does not vanish for $x = x_0$. If, therefore, D is not identically zero, we can express a *general* system of solutions in terms of $y_1, \dots y_4$ and $z_1, \dots z_4$ by means of (2). *We shall, therefore, speak of four pairs of solutions* (y_i, z_i) *for which the determinant* D *does not vanish, as a fundamental system of simultaneous solutions.*

We may express the condition $D \neq 0$ in another way. If $D = 0$, it is possible to find four functions $\lambda, \mu, \nu, \varrho$ of x, so that the four equations

$$
(5) \qquad \lambda y_k + \mu y_k' + \nu z_k + \varrho z_k' = 0 \quad (k = 1, 2, 3, 4)
$$

may be verified

If (y_k, z_k) *form a fundamental system of solutions, it must therefore be impossible to find functions* $\lambda, \mu, \nu, \varrho$ *so as to satisfy* (5), or what amounts to the same thing, it must be impossible to find functions $\alpha, \beta, \gamma, \delta$ of x, which satisfy the system of equations

$$
(6) \qquad \begin{aligned} \alpha y_1 + \beta y_2 + \gamma y_3 + \delta y_4 &= 0, \\ \alpha y_1' + \beta y_2' + \gamma y_3' + \delta y_4' &= 0, \\ \alpha z_1 + \beta z_2 + \gamma z_3 + \delta z_4 &= 0, \\ \alpha z_1' + \beta z_2' + \gamma z_3' + \delta z_4' &= 0. \end{aligned}
$$

In particular $y_1, \dots y_4$ and $z_1, \dots z_4$ must not satisfy two conditions of the form

(7) $$\sum_{i=1}^{4} c_i y_i = 0, \quad \sum_{i=1}^{4} c_i z_i = 0,$$

where $c_1, \dots c_4$ are constants, the same in both equations.

Suppose now that four pairs of functions (y_i, z_i), which verify no relations of the form (5) or (6), are given. Then we may always determine a system of differential equations of the form (A), of which these functions form a simultaneous fundamental system. The coefficients of this system may be obtained from the eight equations (1) by solving for p_{ik} and q_{ik}. If we write

(8) $$D(a_k, b_k, c_k, d_k) = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

we shall find

(9) $$\begin{aligned} Dp_{11} &= -D(y_k'', z_k', y_k, z_k), & Dp_{12} &= -D(y_k', y_k'', y_k, z_k), \\ Dp_{21} &= -D(z_k'', z_k', y_k, z_k), & Dp_{22} &= -D(y_k', z_k'', y_k, z_k), \\ Dq_{11} &= -D(y_k', z_k', y_k'', z_k), & Dq_{12} &= -D(y_k', z_k', y_k, y_k''), \\ Dq_{21} &= -D(y_k', z_k', z_k'', z_k), & Dq_{22} &= -D(y_k', z_k', y_k, z_k''), \end{aligned}$$
$$D = D(y_k', z_k', y_k, z_k)$$

From these equations we find

(10) $$p_{11} + p_{22} = -\frac{1}{D}\frac{dD}{dx},$$

whence

(11) $$D = Ce^{-\int (p_{11}+p_{22})dx},$$

if C denotes a non-vanishing constant.

If we subject the general solutions of system (A) to a transformation of the form

(12) $$\eta = \alpha y + \beta z, \quad \zeta = \gamma y + \delta z,$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x, then η and ζ will be the general solutions of a system of equations of the same form as (A). Moreover, if we put

(13) $$\eta_i = \alpha y_i + \beta z_i, \quad \zeta_i = \gamma y_i + \delta z_i, \quad (i = 1, 2, 3, 4),$$

the four pairs of functions (η_i, ζ_i) will form a fundamental system of solutions for the new system of equations, and its general solutions will be

$$\eta = \sum_{i=1}^{4} c_i \eta_i, \quad \zeta = \sum_{i=1}^{4} c_i \zeta_i.$$

Thus, if we consider instead of a pair of general solutions of (A), four pairs of solutions which form a fundamental system, these are transformed cogrediently with each other and with the pair of general solutions.

§ 2. Geometrical interpretation. The integrating ruled surface of (A).

Let us interpret $(y_1, \ldots y_4)$ and $(z_1, \ldots z_4)$ as the homogeneous coordinates of two points P_y and P_z of space. If (A) is integrated, we shall have these quantities expressed as functions of x:

$$y_k = f_k(x), \; z_k = g_k(x) \quad (k = 1, 2, 3, 4).$$

As x changes, P_y and P_z will describe two curves C_y and C_z; the points of these curves, moreover, are put into a definite correspondence with one another, those being corresponding points which belong to the same value of x.

But there is a restriction on the point-correspondence between these curves, owing to the condition that (y_i, z_i) are to be members of a fundamental system, so that equations (5) must not be verified. Let us write (5) as follows

$$\lambda y_k + \mu y_k' = -(\nu z_k + \varrho z_k') \quad (k = 1, 2, 3, 4).$$

We have seen in Chapter II, § 6 that the quantities

$$\lambda y_k + \mu y_k' \quad (k = 1, 2, 3, 4)$$

represent the homogeneous coordinates of a point on the tangent of the curve C_y at P_y. Similarly $\nu z_k + \varrho z_k'$ are the coordinates of a point on the tangent constructed to C_z at P_z. But, if the above equations hold, these two points coincide, i. e. the two tangents intersect for all values of x.

In order, then, that the curves C_y and C_z may be the integral curves of a system of form (A), *it is necessary and sufficient that the tangents of these two curves, constructed at corresponding points, shall not intersect.*

In particular, the two curves may be plane curves but they must be in different planes.

What is the geometrical significance of the transformations (12) or (13)? Let us mark on the two curves C_y and C_z the points P_y and P_z corresponding to the same value of x, and let us join them by a straight line L_{yz}. Then, it is clear that the transformations (13) convert the points P_y and P_z of the line L_{yz} into two other points P_η and P_ζ of the same line. Consider the ruled surface S, which is the locus of the lines L_{yz} as x passes through all of its values. Since $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x, this transformation enables us to convert the curves C_y and C_z into any other two curves C_η and C_ζ upon this ruled surface. The correspondence of the points

P_η and P_ζ remains such that the line joining corresponding points is a generator of the ruled surface S.

A transformation of the form

$$\xi = f(x),$$

where $f(x)$ is an arbitrary function, changes the parametric representation of the curves in the most general way, without changing either the curves themselves or their point to point correspondence.

Thus, there belongs to every system of two linear homogeneous differential equations of the second order a ruled surface, which we shall call the integrating ruled surface, whose generators are the lines joining corresponding points of the two integral curves. This ruled surface is the same for all such systems which can be transformed into each other by a transformation of the form

$$\begin{aligned} \eta &= \alpha y + \beta z, \\ \zeta &= \gamma y + \delta z, \end{aligned} \quad \xi = f(x),$$

where α, β, γ, δ, f are arbitrary functions of x.

There is one important restriction however; if (y_i, z_i) constitute a fundamental system D must not vanish, i. e. corresponding tangents of the two curves C_y and C_z must not be coplanar. Suppose that they were, so that four functions λ, μ, ν, ϱ might be found which satisfy the equations (5). These equations may be written

$$\frac{d}{dx}(\mu y_k + \varrho z_k) = (\mu' - \lambda)\, y_k + (\varrho' - \nu)\, z_k.$$

But $\mu y_k + \varrho z_k$ are the coordinates u_k of a point P_u of L_{yz}. The right member gives a second point P_v upon L_{yz}, whose coordinates are v_k. As x changes P_u describes a curve C_u, whose tangents, as the above equation shows, are the lines L_{yz}. In this case, therefore, the ruled surface is a developable, since its generators are the tangents of a certain space curve. If, in particular, P_v coincides with P_u we shall have

$$\mu' - \lambda = \omega \mu, \quad \varrho' - \nu = \omega \varrho,$$

where ω is a proportionality factor, i. e.

$$\mu' \varrho - \mu \varrho' - (\lambda \varrho - \mu \nu) = 0.$$

In this case, as x changes, the point P_u remains fixed, i. e. the curve C_u degenerates into a point, and the developable into a cone. We have found the following result.

The integrating ruled surface of a system of form (A) *cannot be a developable. This is the meaning of the condition $D \neq 0$.*

Incidentally we have found a further result, which will be useful later. *If four pairs of functions (y_k, z_k) satisfy a system of equations of the form*

(14) $$\lambda y_k + \mu y_k' + \nu z_k + \varrho z_k' = 0, \quad (k = 1, 2, 3, 4),$$

and if y_k and z_k are interpreted as the homogeneous coordinates of two points P_y and P_z, the line, joining corresponding points of the two curves, generates a developable; the quantities

(15) $$u_k = \mu y_k + \varrho z_k \quad (k = 1, 2, 3, 4)$$

are the coordinates of a point P_u of its cuspidal edge, the developable being the locus of the tangents of the curve C_u described by P_u. If

(16) $$\mu' \varrho - \mu \varrho' - (\lambda \varrho - \mu \nu) = 0,$$

C_u degenerates into a point, and the developable into a cone whose vertex is C_u.

While we shall assume that D is not zero for all values of x, it may happen that for some particular value $x = a$,

$$D = 0.$$

In the vicinity of a generator of this kind, the ruled surface resembles a developable, in so far as all of its tangents along such a generator are coplanar. We may say that, in this case, two consecutive generators of the ruled surface intersect, and speak of the generator as a *torsal generator*. More accurately the state of affairs may be described as follows. If we consider the generators g and g' corresponding to values of x which differ from each other by an infinitesimal Δx, the shortest distance between g and g' will be, in general, an infinitesimal of the order Δx. If it is of a higher order, we may say briefly that the two consecutive generators intersect. Their point of intersection is called a *pinch point* of the surface. Both of these names are due to *Cayley*.[1]) The equation $D = 0$, therefore, characterizes those values of x to which correspond torsal generators of the ruled surface.

But this must be taken with a proviso. Since $y_1 \ldots y_4$ are the homogeneous coordinates of a point, it is not admissible that they should vanish or become infinite simultaneously. For, then the point would be indeterminate. We may therefore express our result as follows.

If for a certain value of $x = a$, neither $y_1, \ldots y_4$ nor $z_1, \ldots z_4$ become simultaneously zero or infinite, while the determinant D vanishes, the corresponding generator of the ruled surface is a torsal generator.

After these remarks it becomes a simple matter to understand an apparent paradox which presents itself in this connection. We have seen in § 1 that

(17) $$D = Ce^{-\int (p_{11} + p_{22})\, dx}.$$

1) *Cayley*'s principal papers on ruled surfaces are in the Cambridge and Dublin Math. Jour. vol. 7 (1852), Phil. Trans. (1863 and 1864), Messenger of Math. 2d. Series, vol. 12 (1882).

But we have also seen in the preceding chapter, that the transformation

(18) $$y = \eta e^{-\frac{1}{2}\int p_{11}\,dx}, \quad z = \zeta e^{-\frac{1}{2}\int p_{22}\,dx}$$

will convert system (A) into another of the same form for which $p_{11} = p_{22} = 0$. This transformation would therefore reduce D to a mere constant. But, if D is a constant it cannot vanish for any value of x, unless it is zero identically The torsal generators seem to be lost in this process, and this constitutes the paradox just mentioned. In the light of our previous theorem this becomes quite clear. If neither $y_1, \ldots y_4$ nor $z_1, \ldots z_4$ are simultaneously zero or infinite for $x = a$, while $D = 0$, the exponent in (17) must be infinite. Therefore, one or both of the exponents in (18) will be infinite, and either $\eta_1, \ldots \eta_4$ or $\zeta_1, \ldots \zeta_4$ or both sets of coordinates will be indeterminate for $x = a$. Strictly speaking, therefore, a transformation of the form (18) is legitimate only for such a portion of a ruled surface as contains no torsal generator.

We return to the consideration of the ruled surface S. This surface has been defined, starting from a particular simultaneous fundamental system of solutions. Since the members of any other fundamental system $(\bar{y}_k, \bar{z}_k)$ can clearly be expressed in the form

$$\bar{y}_k = \sum_{i=1}^{4} c_{ki} y_i, \quad \bar{z}_k = \sum_{i=1}^{4} c_{ki} z_i, \quad (k = 1, 2, 3, 4)$$

any surface obtained from S by a projective transformation may be regarded as integrating ruled surface of (A) as well as S

By means of equations (13) we associated, with each fundamental system of the original system of equations (A), a fundamental system of the transformed system of equations. It is only for such associated fundamental systems of the two sets of equations that it is true that they give rise to the same ruled surface. In general the two integrating ruled surfaces will be merely projective transformations of each other. Let us speak of two systems of differential equations as *equivalent*, if they can be transformed into each other by a transformation of the form (12). We may then state our theorem more precisely as follows:

If two systems of differential equations of form (A) *are equivalent, their integrating ruled surfaces are projective transformations of each other. Moreover if the fundamental systems of solutions be properly selected, the ruled surfaces coincide.*

Conversely, if the ruled surfaces of two such systems coincide, the systems are equivalent.

If the ruled surface is not of the second order, this converse is clear at once. For, the arbitrary functions α, β, γ, δ in the trans-

formation can be chosen so as to convert any pair of curves on the surface, which are not generating straight lines, into any other pair. Thus the pair of curves corresponding to the first system may be converted into the pair corresponding to the second. The independent variables of the two systems of equations will also be expressible in terms of each other, since to every generator of the surface corresponds a value of ξ as well as a value of x. The equivalence of the two systems of equations is therefore ensured, if the ruled surface is not of the second order. In case of a surface of the second order, this conclusion appears doubtful because such a surface contains two distinct sets of generators, and it may happen that the lines joining corresponding points of C_y and C_z are the generators of one set, while those joining C_η and C_ζ are the generators of the second set. In that case the relation between x and ξ cannot be established as before. Since, however, it is always possible to transform the generators of one set into those of the other, by a projective transformation, the theorem is true also if S is a quadric.

Let any non-developable ruled surface S be given. There corresponds to it a class of mutually equivalent systems of linear differential equations. In fact it is easy to indicate how a representative of this class may be found. Trace any two curves, which are not generating straight lines, upon the surface. Express the coordinates of their points as functions of a parameter x, in such a way that to the same value of x correspond points of the two curves which are situated upon the same generator. The system of differential equations whose coefficients may now be found from (9), will have S as integrating ruled surface, and is a representative of the class defined by the surface S.

Therefore, *any non-developable ruled surface may be defined by a system of form* (A). The general theory of such systems of differential equations is, therefore, equivalent to the general theory of ruled surfaces

Any equation or system of equations between p_{ik}, q_{ik}, p'_{ik} etc., which remains invariant for all transformations of the form (12), *expresses a projective property of the integrating ruled surface.*

For, such equations remain unchanged whatever may be the two curves C_y and C_z upon S which are taken as fundamental curves, and whatever may be the independent variable x. They express, therefore, properties of the surface itself, independent of any special method of representation. These properties are projective, because the coefficients of (A) are left invariant by any projective transformation. Conversely, any projective property of a ruled surface can be expressed by an invariant equation, or system of equations.

§ 3. Dualistic interpretation. The adjoint system of (A).

Instead of interpreting $(y_1, \dots y_4)$ and $(z_1, \dots z_4)$ as coordinates of two points, we may consider them as being the coordinates of two planes p_y and p_z. As x changes, these planes envelop two developables D_y and D_z. Corresponding planes intersect along a straight line L_{yz} whose locus forms a ruled surface S, which is left invariant by all of the transformations here considered.

We may combine the two interpretations. At corresponding points P_y and P_z of the two fundamental curves C_y and C_z, let us construct the planes p_y and p_z which are tangent to the ruled surface S. These will intersect along the straight line L_{yz} which joins P_y and P_z. The four pairs of coordinates, determining these planes p_y and p_z, will form a simultaneous fundamental system of solutions for a new system of differential equations, which we shall now deduce and which shall be designated as the *adjoint* of (A). These two systems of differential equations, (A) and its adjoint, will correspond to each other by the principle of duality. It is in this way that we generalize the *Lagrange* adjoint of a single linear differential equation.

Let

$$D = \begin{vmatrix} y_1' & z_1' & y_1 & z_1 \\ y_2' & z_2' & y_2 & z_2 \\ y_3' & z_3' & y_3 & z_3 \\ y_4' & z_4' & y_4 & z_4 \end{vmatrix},$$

so that

$$D = \sum_{k=1}^{4} y_k' v_k = - \sum_{k=1}^{4} z_k' u_k, \tag{19}$$

where

$$\begin{aligned} u_1 &= +(y_2' y_3 z_4), \quad u_2 = -(y_1' y_3 z_4), \quad u_3 = +(y_1' y_2 z_4), \quad u_4 = -(y_1' y_2 z_3), \\ v_1 &= +(z_2' y_3 z_4), \quad v_2 = -(z_1' y_3 z_4), \quad v_3 = +(z_1' y_2 z_4), \quad v_4 = -(z_1' y_2 z_3), \end{aligned} \tag{20}$$

the symbol

$$(a_i b_j c_k)$$

denoting a determinant of the third order whose main diagonal is $a_i b_j c_k$.

It is evident that the homogeneous coordinates of the planes tangent to the integrating ruled surface of system (A) at the points (y_i) and (z_i) respectively, are proportional to (u_i) and (v_i) (cf. Chapter II, § 6).

We shall now prove the following theorem: *If the two fundamental curves C_y and C_z, on the integrating ruled surface S are transformed into two other curves $C_{\bar{y}}$ and $C_{\bar{z}}$ on the surface, by the transformation*

(21) $y_k = \alpha(x)\bar{y}_k + \beta(x)\bar{z}_k, \quad z_k = \gamma(x)\bar{y}_k + \delta(x)\bar{z}_k \quad (k = 1, 2, 3, 4),$

the developable surfaces formed by the planes tangent to the surface S along C_y and C_z are transformed by the equations

(22) $$u_k = \Delta[\alpha(x)\bar{u}_k + \beta(x)\bar{v}_k], \quad v_k = \Delta[\gamma(x)\bar{u}_k + \delta(x)\bar{v}_k] \quad (k = 1, 2, 3, 4),$$

where

(22a) $$\Delta = \alpha\delta - \beta\gamma,$$

i. e., except for the factor Δ, by cogredient transformations.

Proof. We have, the summation everywhere being for $k = 1, 2, 3, 4$,

(23) $$\begin{aligned} &\Sigma u_k z_k' = -D, \quad \Sigma u_k y_k' = 0, \quad \Sigma u_k z_k = 0, \quad \Sigma u_k y_k = 0, \\ &\Sigma v_k z_k' = 0, \quad \Sigma v_k y_k' = D, \quad \Sigma v_k z_k = 0, \quad \Sigma v_k y_k = 0. \end{aligned}$$

Moreover it is clear that these eight equations are just sufficient to determine u_k and v_k. The values (21) of y_k and z_k being introduced, the following system, of which again u_k, v_k are the unique solutions, is obtained:

(24) $$\begin{aligned} &\Sigma u_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = -D, \quad \Sigma u_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma u_k(\alpha\bar{y}_k' + \beta\bar{z}_k' + \alpha'\bar{y}_k + \beta'\bar{z}_k) = 0, \quad \Sigma u_k(\alpha\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma v_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = 0, \quad \Sigma v_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma v_k(\alpha\bar{y}_k' + \beta\bar{z}_k' + \alpha'\bar{y}_k + \beta'\bar{z}_k) = D, \quad \Sigma v_k(\alpha\bar{y}_k + \beta\bar{z}_k) = 0. \end{aligned}$$

By direct computation we find

(25) $$D = \Delta^2\bar{D}.$$

Moreover we have the relations (23) between the transformed quantities $\bar{D}, \bar{u}_k, \bar{v}_k, \bar{y}_k, \bar{z}_k$, i. e.,

(26) $$\begin{aligned} &\Sigma\bar{u}_k\bar{z}_k' = -\bar{D} = -D/\Delta^2, \quad \Sigma\bar{u}_k\bar{y}_k' = 0, \quad \Sigma\bar{u}_k\bar{z}_k = 0, \quad \Sigma\bar{u}_k\bar{y}_k = 0, \\ &\Sigma\bar{v}_k\bar{z}_k' = 0, \quad \Sigma\bar{v}_k\bar{y}_k' = \bar{D} = D/\Delta^2, \quad \Sigma\bar{v}_k\bar{z}_k = 0, \quad \Sigma\bar{v}_k\bar{y}_k = 0. \end{aligned}$$

Multiplying the first four equations of this set in order by $\delta, \gamma, \delta', \gamma'$ and adding, we find the first equation of the following system:

(27) $$\begin{aligned} &\Sigma\bar{u}_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = -\delta D/\Delta^2, \quad \Sigma\bar{u}_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma\bar{u}_k(\alpha\bar{y}_k' + \beta\bar{z}_k' + \alpha'\bar{y}_k + \beta'\bar{z}_k) = -\beta D/\Delta^2, \quad \Sigma\bar{u}_k(\alpha\bar{y}_k + \beta\bar{z}_k) = 0, \\ &\Sigma\bar{v}_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = \gamma D/\Delta^2, \quad \Sigma\bar{v}_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma\bar{v}_k(\alpha\bar{y}_k' + \beta\bar{z}_k' + \alpha'\bar{y}_k + \beta'\bar{z}_k) = \alpha D/\Delta^2, \quad \Sigma\bar{v}_k(\alpha\bar{y}_k + \beta\bar{z}_k) = 0. \end{aligned}$$

From (27) follows very easily a system of precisely the same form as (24), only with

$$\Delta(\alpha\bar{u}_k + \beta\bar{v}_k) \text{ and } \Delta(\gamma\bar{u}_k + \delta\bar{v}_k)$$

in place of u_k and v_k. But equations (24) were sufficient to determine u_k and v_k completely. Therefore we must have

$$u_k = \Delta(\alpha \bar{u}_k + \beta \bar{v}_k), \quad v_k = \Delta(\gamma \bar{u}_k + \delta \bar{v}_k) \quad (k = 1, 2, 3, 4),$$

which proves the theorem.

If we put

$$U_k = \frac{u_k}{\sqrt{D}}, \quad V_k = \frac{v_k}{\sqrt{D}} \quad (k = 1, 2, 3, 4), \tag{28}$$

where the sign of the square root may be chosen at will, (U_k, V_k) are absolutely cogredient with (y_k, z_k).

For $x =$ const. we single out a generator of the ruled surface S and we consider two points P_y and P_z upon it, together with the planes tangent to S at these points. The transformation (21) then transforms P_y and P_z and their tangent planes into $P_{\bar{y}}$ and $P_{\bar{z}}$ and their tangent planes. The transformation (21) is now a linear transformation, and from the fact that the tangent planes are transformed by the cogredient transformations (22) follows the well-known theorem, known as *Chasle*'s correlation, that *the anharmonic ratio of four points on any generator is the same as the anharmonic ratio of the corresponding tangent planes.*

If y_k and z_k form a fundamental system of solutions of equations (A) the determinant D does not vanish, i. e., the ruled surface S is a non-developable surface. If the corresponding determinant for u_k and v_k be formed, its value turns out to be D^3, and therefore also different from zero.

We may therefore regard u_k and v_k as constituting a simultaneous fundamental system of solutions of a pair of equations of the same form as (A). We proceed to set up this system of differential equations.

Denote the minors of z_k and y_k in D by ζ_k and η_k respectively, so that

$$D = \sum_{k=1}^{4} y_k \eta_k = \sum_{k=1}^{4} z_k \zeta_k, \tag{29}$$

where

$$\begin{aligned} \eta_1 &= +(y_2' z_3' z_4), \quad \eta_2 = -(y_1' z_3' z_4), \quad \eta_3 = +(y_1' z_2' z_4), \quad \eta_4 = -(y_1' z_2' z_3), \\ \zeta_1 &= -(y_2' z_3' y_4), \quad \zeta_2 = +(y_1' z_3' y_4), \quad \zeta_3 = -(y_1' z_2' y_4), \quad \zeta_4 = +(y_1' z_2' y_3). \end{aligned} \tag{30}$$

Then, making use of (A), we shall find

$$\begin{aligned} u_k' &= \zeta_k - p_{11} u_k - p_{12} v_k, \\ v_k' &= -\eta_k - p_{21} u_k - p_{22} v_k \end{aligned} \quad (k = 1, 2, 3, 4), \tag{31}$$

whence

$$
(32)\qquad
\begin{aligned}
u_k'' &= \zeta_k' - p_{11}\zeta_k + p_{12}\eta_k + (p_{11}^2 + p_{12}p_{21} - p_{11}')u_k \\
&\qquad + \{(p_{11}+p_{22})p_{12} - p_{12}'\}v_k, \\
v_k'' &= -\eta_k' - p_{21}\zeta_k + p_{22}\eta_k + \{(p_{11}+p_{22})p_{21} - p_{21}'\}u_k \\
&\qquad + (p_{22}^2 + p_{12}p_{21} - p_{22}')v_k.
\end{aligned}
$$

By differentiating (30) we find

$$
\begin{aligned}
\eta_k' &= -(p_{11}+p_{22})\eta_k + q_{11}v_k - q_{21}u_k, \\
\zeta_k' &= -(p_{11}+p_{22})\zeta_k + q_{12}v_k - q_{22}u_k,
\end{aligned}
$$

and from (31)

$$
\begin{aligned}
\eta_k &= -v_k' - p_{21}u_k - p_{22}v_k, \\
\zeta_k &= u_k' + p_{11}u_k + p_{12}v_k.
\end{aligned}
$$

Substituting these values in (32) we find that (u_k, v_k) *are simultaneous solutions of the following system of equations:*

$$
(33)\qquad
\begin{aligned}
u'' + (2p_{11}+p_{22})u' + p_{12}v' &+ \{p_{11}' + q_{22} + (p_{11}+p_{22})p_{11}\}u \\
&+ \{p_{12}' - q_{12} + (p_{11}+p_{22})p_{12}\}v = 0, \\
v'' + p_{21}u' + (2p_{22}+p_{11})v' &+ \{p_{21}' - q_{21} + (p_{11}+p_{22})p_{21}\}u \\
&+ \{p_{22}' + q_{11} + (p_{11}+p_{22})p_{22}\}v = 0.
\end{aligned}
$$

Moreover (u_k, v_k) *form a simultaneous fundamental system of* (33), since, as we have already seen, their determinant does not vanish.

We shall prefer, in general, to use another system, namely, that one whose solutions are the functions U_k and V_k defined by equations (28). Remembering that

$$
(34)\qquad D = Ce^{-\int (p_{11}+p_{22})\,dx},
$$

one sees that this other system may be obtained from (33) by making the transformation

$$
u = Ue^{-\frac{1}{2}\int (p_{11}+p_{22})\,dx}, \qquad v = Ve^{-\frac{1}{2}\int (p_{11}+p_{22})\,dx}.
$$

The resulting system is

$$
(35)\qquad
\begin{aligned}
U'' + p_{11}U' + p_{12}V' + \{q_{11} + \tfrac{1}{4}(u_{11}-u_{22})\}U + \left(q_{12} + \tfrac{1}{2}u_{12}\right)V &= 0 \\
V'' + p_{21}U' + p_{22}V' + \left(q_{21} + \tfrac{1}{2}u_{21}\right)U + \{q_{22} + \tfrac{1}{4}(u_{22}-u_{11})\}V &= 0,
\end{aligned}
$$

where u_{ik} are the same as the quantities so denoted previously [cf. Chapter IV, equations (20)].

A third form, which may be convenient, is obtained by putting

$$
u = \lambda e^{-\int (p_{11}+p_{22})\,dx}, \quad v = \mu e^{-\int (p_{11}+p_{22})\,dx}.
$$

Its fundamental solutions are

$$
\lambda_k = \frac{u_k}{D}, \quad \mu_k = \frac{v_k}{D} \quad (k = 1, 2, 3, 4).
$$

The equations which λ and μ satisfy are

$$(36)\qquad \begin{aligned} &\lambda'' - p_{22}\lambda' + p_{12}\mu' - (p'_{22} - q_{22})\lambda + (p'_{12} - q_{12})\mu = 0, \\ &\mu'' + p_{21}\lambda' - p_{11}\mu' + (p'_{21} - q_{21})\lambda - (p'_{11} - q_{11})\mu = 0. \end{aligned}$$

We shall speak of system (35) *as the system adjoined to* (A). Of course (33) and (36) have essentially the same properties as (35). But the relation of (35) to (A) is somewhat simpler because its solutions are absolutely cogredient with the solutions of (A) under transformation (21), while those of (33) and (36) are cogredient with (A) except for a factor.

§ 4. Properties of adjoined systems. Reciprocity.

The relation of adjoined systems to each other is a very close one. In the first place *they have the same seminvariants and invariants.*

For, if we form the quantities u_{ik}, v_{ik}, w_{ik} for the system (35) and denote them by capital letters, we find

$$(37)\qquad U_{11} = u_{22}, \quad U_{12} = -u_{12}, \quad U_{21} = -u_{21}, \quad U_{22} = u_{11},$$

and similarly for V_{ik} and W_{ik}.

The relation between systems (A) *and* (35) *is a reciprocal one, i.e., if of two systems the second is the adjoined of the first, then the first is also the adjoined of the second.*

For let us denote the coefficients of (35) by P_{ik} and Q_{ik}. Then

$$(38)\qquad \begin{aligned} &P_{ik} = p_{ik}, \\ &Q_{11} = q_{11} + \tfrac{1}{4}(u_{11} - u_{22}), \quad Q_{12} = q_{12} + \tfrac{1}{2}u_{12}, \\ &Q_{22} = q_{22} + \tfrac{1}{4}(u_{22} - u_{11}), \quad Q_{21} = q_{21} + \tfrac{1}{2}u_{21}. \end{aligned}$$

But this gives, on account of (37),

$$\begin{aligned} &p_{ik} = P_{ik}, \\ &q_{11} = Q_{11} + \tfrac{1}{4}(U_{11} - U_{22}), \quad q_{12} = Q_{12} + \tfrac{1}{2}U_{12}, \\ &q_{22} = Q_{22} + \tfrac{1}{4}(U_{22} - U_{11}), \quad q_{21} = Q_{21} + \tfrac{1}{2}U_{21}, \end{aligned}$$

i. e., p_{ik} and q_{ik} are formed from P_{ik} and Q_{ik} just as P_{ik} and Q_{ik} are formed from p_{ik} and q_{ik}. This proves the reciprocity of the two systems.

From (38) it will be noted that *the adjoint system coincides with the original, if and only if*

$$(39)\qquad u_{11} - u_{22} = u_{12} = u_{21} = 0,$$

the meaning of which system of equations we shall find as an immediate consequence of this remark.

Since $-u_k$ and v_k are the minors of z_k' and y_k' respectively, in D, we have

$$\sum_{i=1}^{4} U_i y_i = 0, \quad \sum_{i=1}^{4} V_i z_i = 0, \quad \sum_{i=1}^{4} U_i z_i = 0, \quad \sum_{i=1}^{4} V_i y_i = 0.$$

If the differential equations for U and V are the same as those for y and z, we must have

$$U_i = \sum_{k=1}^{4} c_{ik} y_k, \quad V_i = \sum_{k=1}^{4} c_{ik} z_k \quad (k = 1,\ 2,\ 3,\ 4),$$

where c_{ik} are constants. But these expressions, when substituted in the preceding equations, show that the curves C_y and C_z are situated on the same quadratic surface, and that the line joining the points P_y and P_z is a generator of this surface.

We can therefore say: *a system of two linear differential equations of the second order is identical with its adjoined system, if and only if its integrating ruled surface is of the second order.*

If the original system has either the semi-canonical or the canonical form the same is true of the adjoined system.

From our definitions of the quantities involved we have the following relations:

$$(40) \qquad \begin{aligned} &\Sigma z_k' U_k = -\sqrt{D}, && \Sigma y_k' V_k = +\sqrt{D}, \\ &\Sigma y_k U_k = 0, && \Sigma y_k V_k = 0, \\ &\Sigma z_k U_k = 0, && \Sigma z_k V_k = 0, \\ &\Sigma y_k' U_k = 0, && \Sigma z_k' V_k = 0, \end{aligned}$$

where

$$(41) \qquad D = D(y_k',\ z_k',\ y_k,\ z_k).$$

It follows from the reciprocity of the two systems and may also be verified directly by differentiation of (40) that

$$(42) \qquad \begin{aligned} &\Sigma V_k' y_k = -\sqrt{D}, && \Sigma U_k' z_k = +\sqrt{D}, \\ &\Sigma U_k y_k = 0, && \Sigma U_k z_k = 0, \\ &\Sigma V_k y_k = 0, && \Sigma V_k z_k = 0, \\ &\Sigma U_k' y_k = 0, && \Sigma V_k' z_k = 0, \end{aligned}$$

where D in the first place stands for

$$D(U_k'\ V_k',\ U_k,\ V_k);$$

but this is the same as (41), for

$$(43) \qquad D(U_k',\ V_k',\ U_k,\ V_k) = D(y_k',\ z_k',\ y_k,\ z_k).$$

Now let y_k and z_k be simultaneously transformed into

$$\bar{y}_k = \sum_{i=1}^{4} c_{ki} y_i, \quad \bar{z}_k = \sum_{i=1}^{4} c_{ki} z_i \quad (k = 1, 2, 3, 4),$$

where c_{ki} are constants, whose determinant does not vanish. We may look upon such a transformation as a change of the tetrahedron of reference, or else as a projective transformation of the integral curves.

Equations (40) will be satisfied by the transformed quantities as well. Therefore

$$\sum_{i=1}^{4} z_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = -\sqrt{D}, \quad \sum_{i=1}^{4} y_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = +\sqrt{D},$$

$$\sum_{i=1}^{4} y_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = 0, \quad \sum_{i=1}^{4} y_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = 0,$$

$$\sum_{i=1}^{4} z_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = 0, \quad \sum_{i=1}^{4} z_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = 0,$$

$$\sum_{i=1}^{4} y_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = 0, \quad \sum_{i=1}^{4} z_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = 0,$$

where C denotes the determinant of the transformation (44).

These are of exactly the same form as (40), except that U_k and V_k have been replaced by

$$\frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \bar{U}_k, \quad \frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \bar{V}_k$$

respectively. But on the other hand equations (40) are sufficient to determine U_k and V_k as their solutions.

Therefore we must have

$$(45) \quad U_i = \frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \bar{U}_k, \quad V_i = \frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \bar{V}_k \quad (k = 1, 2, 3, 4),$$

or, solving for $\bar{U}_k$ and $\bar{V}_k$,

$$(46) \quad \bar{U}_k = \frac{1}{\sqrt{C}} \sum_{i=1}^{4} C_{ki} U_i, \quad \bar{V}_k = \frac{1}{\sqrt{C}} \sum_{i=1}^{4} C_{ki} V_i \quad (k = 1, 2, 3, 4),$$

where C_{ki} is the minor of c_{ki} in the determinant

$$C = | c_{ki} |.$$

We may state our result as follows: *If the elements y_i and z_i of a simultaneous fundamental system of* (A) *are made to undergo cogredient linear substitutions with constant coefficients and non-vanishing determinants, the corresponding solutions of the adjoined system of differential equations also undergo mutually cogredient linear substitutions with constant coefficients. The coefficients of the second set of substitutions are the minors of those of the first set in their determinant, divided by the square root of that determinant. In a slightly modified sense then, the quantities* (y_k, z_k) *and* (U_k, V_k) *are contragredient. The quantities* (y_k, z_k) *and* $(\lambda_k, \mu_k) = (u_k/D, v_k/D)$ *are contragredient in the ordinary sense of the word.*

Upon this theorem rests the simple relation between the monodromic groups and the transformation groups of reciprocal systems of differential equations.[1])

We may complete the relations (40) in an interesting manner. We have

$$\begin{aligned} y_k'' + p_{11} y_k' + p_{12} z_k' + q_{11} y_k + q_{12} z_k &= 0, \\ z_k'' + p_{21} y_k' + p_{22} z_k' + q_{21} y_k + q_{22} z_k &= 0 \end{aligned} \qquad (k = 1, 2, 3, 4).$$

If we multiply these equations by U_k and V_k successively and add, taking into account the relations (40), we shall find

$$(47) \qquad \begin{aligned} -\sum_{k=1}^{4} y_k' U_k' = \sum_{k=1}^{4} y_k'' U_k &= + p_{12}\sqrt{D}, \\ \sum_{k=1}^{4} y_k'' V_k &= - p_{11}\sqrt{D}, \\ \sum_{k=1}^{4} z_k'' U_k &= + p_{22}\sqrt{D}, \\ -\sum_{k=1}^{4} z_k' V_k' = \sum_{k=1}^{4} z_k'' V_k &= - p_{21}\sqrt{D}. \end{aligned}$$

Also, if (40) and (47) are used,

$$(48) \qquad \sum_{k=1}^{4} y_k' V_k' = \frac{D'}{2\sqrt{D}} + p_{11}\sqrt{D}, \quad \sum_{k=1}^{4} z_k' U_k' = -\frac{D'}{2\sqrt{D}} - p_{22}\sqrt{D}.$$

---

1) We shall have no occasion in this work to discuss these notions. They may be defined by generalizing the corresponding concepts for a single linear differential equation.

Finally, treating (35) as we have just treated (A), we obtain

$$(49)\qquad \begin{aligned} \sum_{k=1}^{4} U_k'' y_k &= +p_{12}\sqrt{D}, & \sum_{k=1}^{4} U_k'' z_k &= -p_{11}\sqrt{D}, \\ \sum_{k=1}^{4} V_k'' y_k &= +p_{22}\sqrt{D}, & \sum_{k=1}^{4} V_k'' z_k &= -p_{21}\sqrt{D}, \end{aligned}$$

as may also be seen from the reciprocity of (A) and (35).

We have seen in Chapter IV that every system of form (A) may be reduced to a *semi-canonical* form characterized by the conditions $p_{ik} = 0$. We can now show that this transformation corresponds to the determination of the asymptotic curves of the integrating ruled surface.[1])

As a matter of fact a simpler reduction is sufficient to determine the asymptotic lines. For, let the given system be transformed into another for which merely $p_{12} = p_{21} = 0$, while p_{11} and p_{22} may be arbitrary. Then the integral curves C_y and C_z on S will be such that

$$\begin{aligned} \sum_{k=1}^{4} U_k y_k &= 0, & \sum_{k=1}^{4} U_k y_k' &= 0, & \sum_{k=1}^{4} U_k y_k'' &= 0, \\ \sum_{k=1}^{4} V_k z_k &= 0, & \sum_{k=1}^{4} V_k z_k' &= 0, & \sum_{k=1}^{4} V_k z_k'' &= 0, \end{aligned}$$

i. e, the plane tangent to S at (y_1, y_2, y_3, y_4) is the osculating plane of the curve C_y at that point, and the plane tangent to S at (z_1, z_2, z_3, z_4) is the osculating plane of C at that point Therefore C_y and C_z are asymptotic lines of the surface, asymptotic curves of the *second set,* those of the first set being the generators of the surface.

If, then, in any system of form (A), $p_{12} = p_{21} = 0$, *its integral curves are asymptotic lines on its integrating ruled surface.*

If a given system (A) has by a first transformation been converted into another for which $p_{12} = p_{21} = 0$, the semi-canonical form for which p_{11} and p_{22} also vanish, may be obtained very easily by putting

$$y = \eta e^{-\frac{1}{2}\int p_{11} dx}, \quad z = \zeta e^{-\frac{1}{2}\int p_{22} dx}.$$

Since such a transformation merely multiplies $y_1, \ldots, y_4$ by the same factor, and similarly $z_1, \ldots, z_4$, it does not affect the significance of these quantities as the homogeneous coordinates of corresponding points on two asymptotic lines.

---

1) A curve is an asymptotic line upon a surface, if its osculating plane, at each of its points, coincides with the plane tangent to the surface at that point. A surface has two families of asymptotic curves upon it, which coincide only if the surface is a developable

If the system (A) is written in the semi-canonical form, the most general transformation which leaves this form invariant was found to be [Chapter IV, equations (90)]

$$\xi = \xi(x), \quad \eta = \sqrt{\xi'}\,(ay + bz), \quad \zeta = \sqrt{\xi'}\,(cy + dz),$$

where ξ is an arbitrary function of x, and where a, b, c, d are arbitrary constants.

These equations show, in the first place, that there exists upon the ruled surface a single infinity of curved asymptotic lines. But further these equations show that *the double-ratio of the four points in which any generator of the surface intersects four fixed asymptotic curves of the second set, is constant.*

This theorem is due to *Paul Serret*[1]), and gives a most elegant generalization of the well-known property of a quadric ruled surface.

§ 5. The fundamental theorem of the theory of ruled surfaces.

In Chapter IV, § 7, we have shown that, if the invariants Θ_4, $\Theta_{4\cdot 1}$, Θ_9 and Θ_{10} are given as functions of x, provided that Θ_4 and Θ_{10} are not zero, a system of differential equations of the form (A) can be written down whose invariants coincide with these arbitrarily given functions of x. Its coefficients were given by the equations (102) of that paragraph, and all other systems of the form (A) which have the same invariants were found to be equivalent to this special system (102). We may now express this theorem in the following form.

If Θ_4, $\Theta_{4\cdot 1}$, Θ_9 and Θ_{10} are given as arbitrary functions of x, provided however that Θ_4 and Θ_{10} are not identically equal to zero, they determine a ruled surface uniquely except for projective transformations.

This theorem may be regarded as *the fundamental theorem of the theory of ruled surfaces.*

If we denote the invariants of the adjoint system by $\bar{\Theta}_4$, $\bar{\Theta}_{4\cdot 1}$, etc., we find from (37) and (38),

$$(50) \qquad \bar{\Theta}_4 = \Theta_4, \quad \bar{\Theta}_{4\cdot 1} = \Theta_{4\cdot 1}, \quad \bar{\Theta}_9 = -\Theta_9, \quad \bar{\Theta}_{10} = \Theta_{10}.$$

We may, instead of interpreting U_k, V_k as coordinates of planes interpret them as point coordinates. Then, the integrating ruled surface of the adjoint system, instead of being the surface S in plane coordinates, will be a surface S', dualistic to S, in point coordinates. We have, therefore, the following further theorem.

---

1) *P. Serret.* Théorie Nouvelle Géométrique et Mécanique des Ligues à Double Courbure. (Paris-Bachelier, 1860.)

If the fundamental invariants of two ruled surfaces S and S' expressed as functions of the same variable x, satisfy the relations (50), *the two surfaces are dualistic to each other.*

We notice at once a further consequence. The system (A) and its adjoint are referred to the same independent variable x. If (A) and its adjoint are equivalent, it must, therefore, be possible to transform (A) into its adjoint by a transformation

$$U = \alpha y + \beta z, \quad V = \gamma y + \delta z, \tag{51}$$

involving the dependent variables only. But such a transformation leaves the invariants, *absolutely* unchanged. Therefore, (A) and its adjoint can be equivalent only if

$$\Theta_9 = 0.$$

Moreover, as our fundamental theorem shows, if $\Theta_4 \neq 0$, and $\Theta_{10} \neq 0$, this condition $\Theta_9 = 0$, is not only necessary but also sufficient for the equivalence of (A) and its adjoint. As (51) shows, the integrating ruled surface of the adjoint will coincide with S, generator for generator. Let us speak of a ruled surface as being *identically self-dual,* if a dualistic transformation exists, which converts it into itself generator for generator. Then we have seen, that *the necessary and sufficient condition for an identically self-dual ruled surface is $\Theta_9 = 0$, provided that Θ_4 and Θ_{10} are not zero.*

We shall find, later, a very simple interpretation for the condition $\Theta_9 = 0$, which will make the truth of this result intuitively evident.

In the case that

$$u_{11} - u_{22} = u_{12} = u_{21} = 0,$$

the adjoint system coincides with (A), so that we may put

$$U = y, \quad V = z.$$

Therefore, the quadric surface is identically self-dual in a still more special sense. There exists a dualistic transformation which converts it into itself, *point for point.* More strictly speaking, this transformation converts every point of the surface into its tangent plane, and every tangent plane into its point of contact. It is evident what this dualistic transformation is; it is merely the pole-polar transformation with respect to the quadric itself.

We have proved the fundamental theorem under the assumptions $\Theta_4 \neq 0$, $\Theta_{10} \neq 0$. We shall see that theorem actually breaks down if either Θ_4 or Θ_{10} vanishes. We prefer to leave the proof of this statement for a later chapter, as we shall then be able to interpret these conditions geometrically.

Examples.

Ex. 1. The equation of a cubic ruled surface, with distinct directrices, may be written $x_3 x_1^2 - x_4 x_2^2 = 0$. The two curves (straight line directrices),

$$y_1 = y_2 = 0, \quad y_3 = x^2, \quad y_4 = 1; \quad z_1 = 1, \quad z_2 = x, \quad z_3 = z_4 = 0;$$

are upon it. Show that the differential equations of the surface are

$$y'' - \frac{1}{x} y' = 0, \quad z'' = 0.$$

Compute its invariants. Show that its asymptotic curves are unicursal quartics, which intersect every generator in two points harmonic conjugates with respect to its intersections with the directrices.

Ex. 2. If the directrices of the cubic coincide (*Cayley's* cubic scroll), its equation may be written $x_2^3 + x_1(x_1 x_3 + x_2 x_4) = 0$. The curves

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = -x, \quad y_4 = 1; \quad \text{(directrix)},$$
$$z_1 = -1, \quad z_2 = -x, \quad z_3 = 0, \quad z_4 = x^2;$$

are upon it. Its differential equations are

$$y'' = 0, \quad z'' + xy' - y = 0.$$

Its asymptotic curves are twisted cubics. All of its invariants vanish.

Ex. 3. Find the pinch-points of the above surfaces, and show that the asymptotic curves pass through them (*Snyder*).

Ex. 4. If

$$y_1 = 2kl + \frac{3}{2} k^2 x + \frac{3}{2} l x^2 + \frac{3}{4} k x^3,$$

$$y_2 = k^2 + 2lx + \frac{3}{2} k x^2,$$

$$y_3 = l + \frac{3}{2} kx,$$

$$y_4 = k,$$

and

$$z_1 = k + \frac{3}{2} x^2, \quad z_2 = 2x, \quad z_3 = 1, \quad z_4 = 0,$$

the line $P_y P_z$ generates a developable, whose edge of regression is the cubic

$$\beta_1 = l + \frac{3}{2} kx + \frac{3}{4} x^3, \quad \beta_2 = k + \frac{3}{2} x^2, \quad \beta_3 = \frac{3}{2} x, \quad \beta_4 = 1.$$

Ex. 5. For the general transformation, the left members of the adjoint system are cogredient with y and z, except for a power of ξ'. They are absolutely cogredient with the left members of the system (A). Thence deduce a new proof that C is a covariant.

CHAPTER VI.

SIGNIFICANCE OF THE COVARIANTS P AND C.

§ 1. The flecnode curve and the flecnodal surface.

Consider the quantities ϱ and σ defined by the equations (112) of Chapter IV. If we substitute $y = y_k$, $z = z_k$, $(k = 1, 2, 3, 4)$ in these expressions, we obtain

$$(1) \qquad \begin{aligned} \varrho_k &= 2y_k' + p_{11}y_k + p_{12}z_k, \\ \sigma_k &= 2z_k' + p_{21}y_k + p_{22}z_k, \end{aligned} \qquad (k = 1, 2, 3, 4),$$

which quantities may again be interpreted as the homogeneous coordinates of two points P_ϱ and P_σ. Clearly P_ϱ is a point of the plane tangent to the integrating ruled surface S of (A) at P_y, and P_σ is a point of the plane tangent to S at P_z.

If the points P_y and P_z be transformed into two other points $P_{\bar{y}}$, $P_{\bar{z}}$ of the line L_{yz} which joins them, by the equations

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z},$$

then, as has already been noted in Chapter IV, P_ϱ and P_σ will be transformed cogrediently into $P_{\bar\varrho}$ and $P_{\bar\sigma}$, where

$$\varrho = \alpha\bar{\varrho} + \beta\bar{\sigma}, \quad \sigma = \gamma\bar{\varrho} + \delta\bar{\sigma},$$

i. e. into two other points of the line $L_{\varrho\sigma}$ which joins P_ϱ to P_σ.

Thus, we have, by means of equations (1), a straight line $L_{\varrho\sigma}$ corresponding to every generator L_{yz} of S. Moreover, there is a one-to-one correspondence between the points of these two lines, which we now propose to investigate.

For this purpose, suppose that (A) has been reduced to its semi-canonical form, so that C_y and C_z are two asymptotic curves of S, and $p_{ik} = 0$. We shall then have

$$\varrho_k = 2y_k', \quad \sigma = 2z_k',$$

i. e. P_ϱ and P_σ are points upon the tangents of C_y and C_z. If therefore, we consider any point P upon the generator L_{yz} of S, the point P' of the line $L_{\varrho\sigma}$ which corresponds to it, is situated upon the tangent t of the asymptotic curve of S which passes through P. Now, as P moves along the generator L_{yz} or g of S, this tangent t describes a hyperboloid H. For, the asymptotic tangent t of the point P of g is determined by the condition that it shall also intersect g' and g'', two generators of S infinitesimally close to g. The hyperboloid H shall be called the *hyperboloid osculating S along g*. We shall speak of those generators of H which are of the same kind as g, as its *generators of the first kind*. Then, as P moves along g, t coincides successively with all of the generators of the second kind

on H. P' is a point of t, and its locus is a straight line $L_{\varrho\sigma}$. Therefore, $L_{\varrho\sigma}$ can only be a generator of the first kind upon H.

The line $L_{\varrho\sigma}$ is a generator of the first kind upon the hyperboloid osculating the ruled surface along g. The correspondence between the points of g and of $L_{\varrho\sigma}$ is such that the straight lines joining corresponding points are the generators of the second kind upon the osculating hyperboloid.

The position of $L_{\varrho\sigma}$ upon the osculating hyperboloid changes when the independent variable x is transformed. We have seen in chapter IV, equation (115), that if we put

$$\xi = \xi(x),$$

ϱ and σ are converted into $\bar{\varrho}$ and $\bar{\sigma}$, where

$$\bar{\varrho} = \frac{1}{\xi'}(\varrho + \eta y), \quad \bar{\sigma} = \frac{1}{\xi'}(\sigma + \eta z), \quad \eta = \frac{\xi''}{\xi'}.$$

Clearly, η may be chosen in such a way as to make $L_{\varrho\sigma}$ coincide with any generator of the first kind upon H. We shall construct the line $L_{\varrho\sigma}$ for every value of x, and thus get a new ruled surface S' associated with S. We shall speak of S' as the *derivative of S with respect to x*. If S' is given, η is known as function of x, and ξ is determined save for a linear transformation. The derivative ruled surface may, therefore, serve as an image of the independent variable. This image does not change if x is converted into $ax + b$ where a and b are arbitrary constants (cf. Chapter III).

If the independent variable of (A) is given, the generator $L_{\varrho\sigma}$ of the derived surface S' may be defined directly by a limit process. Let us assume $p_{ik} = 0$, so that C_y and C_z are asymptotic curves upon S. Then

$$\varrho = 2y', \quad \sigma = 2z'.$$

Consider the three consecutive generators g_{-1}, g_0, g_{+1} of S as belonging to the values of x,

$$x_0 - \Delta x, \quad x_0, \quad x_0 + \Delta x$$

respectively, where Δx is an infinitesimal. Construct the tangent to C_y at P_y. It meets the three generators g_{-1}, g_0, g_{+1} since C_y is an asymptotic curve. The coordinates of the three points of intersection, A, B and C, are

$$y_k - y_k' \Delta x, \quad y_k, \quad y_k + y_k' \Delta x,$$

so that the point P_ϱ whose coordinates are proportional to y_k', is the harmonic conjugate of B with respect to A and C. Similarly for P_σ. Therefore the line $L_{\varrho\sigma}$ may be selected as follows. We consider three generators of the surface S, corresponding to three values of x forming an arithmetical progression of common difference d. Upon the hyperboloid, determined by these three lines, we construct

the generator which is the harmonic conjugate of the middle line with respect to the other two. As the common difference d approaches the limit zero, the hyperboloid approaches as a limit the osculating hyperboloid, and the fourth generator approaches as a limit the line $L_{\varrho\sigma}$.

Omitting the subscripts in (1), we find by differentiation

$$\varrho' = 2y'' + p_{11}y' + p_{12}z' + p'_{11}y + p'_{12}z,$$
$$\sigma' = 2z'' + p_{21}y' + p_{22}z' + p'_{21}y + p'_{22}z,$$

whence, making use of (A),

$$\varrho' = -p_{11}y' - p_{12}z' + (p'_{11} - 2q_{11})y + (p'_{12} - 2q_{12})z,$$
$$\sigma' = -p_{21}y' - p_{22}z' + (p'_{21} - 2q_{21})y + (p'_{22} - 2q_{21})z.$$

If the values of y' and z' in terms of y, z, ϱ, σ be substituted from (1), the following equations will be obtained:

$$(2) \qquad \begin{aligned} R &= 2\varrho' + p_{11}\varrho + p_{12}\sigma = u_{11}y + u_{12}z, \\ S &= 2\sigma' + p_{21}\varrho + p_{22}\sigma = u_{21}y + u_{22}z, \end{aligned}$$

where R and S are merely abbreviations for the left members, and where the quantities u_{ik} are the same as those which have been previously denoted by these symbols.

The left members of (2), for $\varrho = \varrho_k$, $\sigma = \sigma_k$, are clearly the coordinates of two points, one in the plane tangent to S' at P_ϱ and one in the plane tangent to S' at P_σ. The equations (2) show, therefore, that if the planes tangent to S' at P_ϱ and P_σ are constructed, they will intersect the generator L_{yz} of S in the points $u_{11}y_k + u_{12}z_k$ and $u_{21}y_k + u_{22}z_k$ respectively. Or, in other words, the lines joining ϱ_k with $u_{11}y_k + u_{12}z_k$, and σ_k with $u_{21}y_k + u_{22}z_k$ are tangents of the derivative ruled surface S' at P_ϱ and P_σ respectively

In particular then, *if $u_{12} = u_{21} = 0$, the lines which are tangent to the asymptotic curves of the surface S at P_y and P_z, are also tangents of the derived ruled surface S'.* But we can find a simpler and more fundamental interpretation for the conditions $u_{12} = u_{21} = 0$.

Consider three consecutive generators g_{-1}, g_0, g_{+1} of the ruled surface S. The hyperboloid H_0, osculating S along g_0, is determined by these three lines. On H_0 we have a line $L_{\varrho\sigma}$, or for short h_0, which is the generator of S' corresponding to the generator g_0 of S. Consider a fourth generator g_2 of S, consecutive to g_1. The lines g_0, g_1, g_2 determine the hyperboloid H_1, osculating S along g_1. There is upon it a line h_1 which is the corresponding generator of S'. Any tangent to S' along h_0 must intersect h_0 and h_1. If it is, at the same time, tangent to an asymptotic curve of S at any point of g_0, it must intersect also the lines g_{-1}, g_0, g_1. Such a line must, therefore, intersect the five lines $g_{-1}, g_0, g_1. h_0, h_1$. But since h_0 is on the hyperboloid determined by g_{-1}, g_0, g_1, we may suppress h_0, since any

line intersecting g_{-1}, g_0, g_1 will also intersect h_0. Therefore, we may say that such a line must intersect the four lines g_{-1}, g_0, g_1, h_1. But any line intersecting g_{-1}, g_0, g_1, is a generator of the second kind of the hyperboloid H_0, and any line intersecting g_0, g_1, h_1, is a generator of the second kind on H_1.

Therefore any line, which is tangent to an asymptotic curve of S at a point of g, and which is at the same time tangent to the derivative surface S' at a point of the generator of that surface which corresponds to g, is common to two consecutive osculating hyperboloids of the surface S. Or, in other words, *such a line intersects four consecutive generators of the surface S.*

There are, in general, two such lines, since four lines in space have two real, imaginary or coincident straight line intersectors. In fact, if l_1, l_2, l_3 are three of the given four lines, any line t which intersects them, is a generator of the second kind on the hyperboloid H determined by l_1, l_2, l_3. The line l_4 intersects H in two points. Therefore, the required intersectors of l_1, l_2, l_3, l_4 are those two generators of the second kind on H, which pass through these two points of intersection. They coincide if l_4 is tangent to H.

We have seen in Chapter IV, § 8, that, if the factors of the covariant C be denoted by η and ζ, so that

$$
(3)\qquad
\begin{aligned}
\eta &= \frac{u-\sqrt{\Theta_4}}{2}\,y + u_{12}z = \lambda\left[u_{21}y - \frac{u+\sqrt{\Theta_4}}{2}\,z\right],\\
\zeta &= \frac{u+\sqrt{\Theta_4}}{2}\,y + u_{12}z = \mu\left[u_{21}y - \frac{u-\sqrt{\Theta_4}}{2}\,z\right],
\end{aligned}
$$

where

$$
(4)\quad u = u_{11} - u_{22},\quad \lambda = \frac{u-\sqrt{\Theta_4}}{2u_{21}} = -\frac{2u_{12}}{u+\sqrt{\Theta_4}},\quad \mu = \frac{u+\sqrt{\Theta_4}}{2u_{21}} = -\frac{2u_{12}}{u-\sqrt{\Theta_4}}
$$

the system of differential equations for η and ζ will be of the form (A) and will satisfy the conditions $u_{12} = u_{21} = 0$. Moreover we may see from (44) Chapter IV, that the most general transformation of the dependent variables which leaves the conditions $u_{12} = u_{21} = 0$ invariant, may be compounded from

$$y = \alpha\eta,\quad z = \delta\zeta,$$

and

$$y = \zeta,\quad z = \eta;$$

in other words, the two curves C_η and C_ζ on S are absolutely determined by the conditions $u_{12} = u_{21} = 0$.

Therefore, *the two curves upon S, which are characterized by the conditions* $u_{12} = u_{21} = 0$, *intersect every generator of the surface in the two points at which tangents can be drawn, which have four consecutive points in common with the surface.*

In his general theory of the singularities of a surface[1]), *Cayley* denotes a point, at which a four-point tangent can be drawn, as a *flecnode*. The tangent itself may be called the *flecnode tangent*, and the locus of all of the flecnodes of S, its *flecnode curve*. The ruled surface of two sheets, the locus of the flecnode tangents of S, shall be called its *flecnode surface*.

A flecnode tangent is clearly tangent to the asymptotic curve which passes through the flecnode. It is not, in general, tangent to the flecnode curve. For, if it were, the flecnode curve would be at the same time an asymptotic curve, i. e. if we identified this curve with C_y, we should have simultaneously

$$u_{12} = p_{12} = 0.$$

But from these conditions, we would find that q_{12} also must vanish, so that the first equation of (A) would become

$$y'' + p_{11}y' + q_{11}y = 0.$$

If, however, $y_1, \ldots y_4$ are four solutions of this equation, there must be two homogeneous linear relations, with constant coefficients, between them. For, such an equation can have only two linearly independent solutions. In other words, the curve C_y would be a straight line.

We may recapitulate our main result as follows.

The flecnode curve is determined by factoring the covariant C. Its intersections with the generators of the surface are distinct if $\Theta_4 \neq 0$; they coincide if $\Theta_4 = 0$. If the coefficients of system (A) *are real, and if the solutions y_k and z_k are real, the flecnode curve intersects the generators in real, coincident or imaginary points according as*

$$\Theta_4 \gtreqless 0.$$

If the curves C_y and C_z themselves are the two branches of the flecnode curve, system (A) *is characterized by the conditions*

$$u_{12} = u_{21} = 0.\,^{2})$$

The flecnode tangent is never tangent to a branch of the flecnode curve unless that branch degenerates into a straight line. In that case that branch is also an asymptotic curve, and the corresponding sheet of the flecnode surface degenerates into a straight line.

The flecnode curve becomes indeterminate if

$$u_{11} - u_{22} = u_{12} = u_{21} = 0,$$

---

1) *Cayley*. Mathematical Papers, vol. II, p. 29.

2) If we speak of the two *branches* of the flecnode curve, we must guard against possible misunderstanding of the term. It is merely a word expressing the fact that the curve intersects every generator in two points. The curve need not therefore be a bipartite curve.

i. e. if S is a quadric. That this must be so, is obvious geometrically. But even if $u_{11} - u_{22}, u_{12}, u_{21}$ do not vanish identically, there may be particular values of x for which they do.

For such generators, the flecnodes are indeterminate. We may say that the *osculating hyperboloid hyperosculates the surface*, a singularity of ruled surfaces first mentioned by *Voss* who, it seems, was also the first to consider the flecnode curve.[1])

In this connection we notice further that the covariant C vanishes identically, i. e. for all curves C_y and C_z upon the surface, if and only if the surface is a quadric.

In order to be able to deduce further results from our considerations, it becomes necessary to set up the system of differential equations for the derived ruled surface S'. It will be, of course, a system of form (A) between ϱ and σ.

We find from (2), solving for y and z,

$$
(5) \qquad \begin{aligned} Jy &= \quad u_{22} R - u_{12} S, \\ Jz &= - u_{21} R + u_{11} S, \end{aligned} \qquad J = u_{11} u_{22} - u_{12} u_{21}.
$$

Further, if (2) be differentiated, and if the values of y' and z' be expressed in terms of y, z, ϱ, σ from (1), we shall find

$$
\begin{aligned} 2R' &= u_{11}\varrho + u_{12}\sigma + (2u_{11}' - u_{11}p_{11} - u_{12}p_{21})\,y + (2u_{12}' - u_{11}p_{12} - u_{12}p_{22})\,z, \\ 2S' &= u_{21}\varrho + u_{22}\sigma + (2u_{21}' - u_{21}p_{11} - u_{22}p_{21})\,y + (2u_{22}' - u_{21}p_{12} - u_{22}p_{22})\,z. \end{aligned}
$$

The quantities u_{ik}' may be expressed in terms of the quantities p_{ik} and v_{ik} by equations (32) of Chapter IV. If this be done, if moreover both members of the equations be multiplied by J, and if use be made of (5), these equations become

$$
(6) \qquad \begin{aligned} 2JR' - Ju_{11}\varrho - Ju_{12}\sigma + t_{11}R + t_{12}S &= 0, \\ 2JS' - Ju_{21}\varrho - Ju_{22}\sigma + t_{21}R + t_{22}S &= 0, \end{aligned}
$$

where we have put

$$
(7) \qquad \begin{aligned} t_{11} &= Jp_{11} + u_{21}v_{12} - u_{22}v_{11}, \\ t_{12} &= Jp_{12} - u_{11}v_{12} + u_{12}v_{11}, \\ t_{21} &= Jp_{21} + u_{21}v_{22} - u_{22}v_{21}, \\ t_{22} &= Jp_{22} - u_{11}v_{22} + u_{12}v_{21}. \end{aligned}
$$

Performing the differentiations indicated, inserting the values of R and S from (2), and collecting terms, we find the required system of differential equations:

1) *Voss*. Zur Theorie der windschiefen Flächen, Mathematische Annalen, vol. 8.

$$
(8)\quad
\begin{aligned}
&4J\varrho'' + 2(Jp_{11} + t_{11})\varrho' + 2(Jp_{12} + t_{12})\sigma' + (2Jp'_{11} - Ju_{11} \\
&\qquad + t_{11}p_{11} + t_{12}p_{21})\varrho + (2Jp'_{12} - Ju_{12} + t_{11}p_{12} + t_{12}p_{22})\sigma = 0, \\
&4J\sigma'' + 2(Jp_{21} + t_{21})\varrho' + 2(Jp_{22} + t_{22})\sigma' + (2Jp'_{21} - Ju_{21} \\
&\qquad + t_{21}p_{11} + t_{22}p_{21})\varrho + (2Jp'_{22} - Ju_{22} + t_{21}p_{12} + t_{22}p_{22})\sigma = 0.
\end{aligned}
$$

Let us put

$$
(9)\quad
\begin{aligned}
u_{21}v_{12} - u_{22}v_{11} &= 2J\lambda_{11}, & u_{21}v_{22} - u_{22}v_{21} &= 2J\lambda_{21}, \\
-u_{11}v_{12} + u_{12}v_{11} &= 2J\lambda_{12}, & -u_{11}v_{22} + u_{12}v_{21} &= 2J\lambda_{22},
\end{aligned}
$$

so that

$$
(10)\quad
\begin{aligned}
2(u_{11}\lambda_{11} + u_{21}\lambda_{12}) &= -v_{11}, & 2(u_{11}\lambda_{21} + u_{21}\lambda_{22}) &= -v_{21}, \\
2(u_{12}\lambda_{11} + u_{22}\lambda_{12}) &= -v_{12}, & 2(u_{12}\lambda_{21} + u_{22}\lambda_{22}) &= -v_{22}.
\end{aligned}
$$

We may then write the system (8) as follows

$$
(11)\quad
\begin{aligned}
\varrho'' + P_{11}\varrho' + P_{12}\sigma' + Q_{11}\varrho + Q_{12}\sigma = 0, \\
\sigma'' + P_{21}\varrho' + P_{22}\sigma' + Q_{21}\varrho + Q_{22}\sigma = 0,
\end{aligned}
$$

where

$$P_{ik} = p_{ik} + \lambda_{ik} \quad (i, k = 1, 2),$$

$$
(12)\quad
\begin{aligned}
Q_{11} &= q_{11} + \frac{1}{2}(\lambda_{11}p_{11} + \lambda_{12}p_{21}), & Q_{21} &= q_{21} + \frac{1}{2}(\lambda_{21}p_{11} + \lambda_{22}p_{21}), \\
Q_{12} &= q_{12} + \frac{1}{2}(\lambda_{11}p_{12} + \lambda_{12}p_{22}), & Q_{22} &= q_{22} + \frac{1}{2}(\lambda_{21}p_{12} + \lambda_{22}p_{22}).
\end{aligned}
$$

These equations will be of the greatest importance in a later chapter. Our present purpose, however, is to eliminate $z, z', z'', \sigma, \sigma', \sigma''$ from the eight equations (A), (1), (2) and (11), under the assumption $u_{12} = u_{21} = p_{11} = p_{22} = 0$. We shall thus find a system of differential equations between y and ϱ, whose integrating ruled surface is one sheet, F', of the flecnode surface of S. We may find in the same way the equations for the second sheet, F'', of the flecnode surface.

We assume, therefore, $p_{11} = p_{22} = u_{12} = u_{21} = 0$. We find, from (1),

$$(13)\qquad p_{12}z = \varrho - 2y',$$

whence by differentiation

$$(14)\qquad p_{12}^2 z' = -2p_{12}y'' + 2p'_{12}y' + p_{12}\varrho' - p'_{12}\varrho.$$

If we substitute these values into the first equation of (A), we shall find the equation

$$-p_{12}y'' + 2(p'_{12} - q_{12})y' + p_{12}\varrho' + p_{12}q_{11}y - (p'_{12} - q_{12})\varrho = 0,$$

which is one of the required differential equations.

From (2) we find

$$2\sigma' = -p_{21}\varrho + u_{22}z, \quad p_{12}\sigma = -2\varrho' + u_{11}y.$$

If these values be substituted into the first equation of (11), we shall find

$$
\begin{aligned}
&2p_{12}\varrho'' + 2(p_{12}P_{11} - 2Q_{12})\varrho' - 2P_{12}u_{22}y' + 2u_{11}Q_{12}y \\
&\qquad + (2p_{12}Q_{11} + P_{12}u_{22} - P_{12}p_{12}p_{21})\varrho = 0,
\end{aligned}
$$

which is the second equation of the required system. If the values of P_{11}, Q_{11}, P_{12}, Q_{12} under the special assumptions, $p_{11}=p_{22}=u_{12}=u_{21}=0$, be introduced, we find finally the following system of differential equations for the sheet F' of the flecnode surface of S:

$$
\begin{aligned}
&y'' - 2\frac{q_{12}}{p_{12}}y' - \varrho' - q_{11}y + \frac{q_{12}}{p_{12}}\varrho = 0,\\
(15)\qquad &\varrho'' + [2(q_{11}+q_{22}) - p_{12}p_{21}]y' - 2\frac{q_{12}}{p_{12}}\varrho'\\
&\quad + \left[2q_{11}' - p_{12}q_{21} - 4\frac{q_{12}}{p_{12}}q_{11}\right]y - q_{22}\varrho = 0.
\end{aligned}
$$

In precisely the same way we find the equations of the second sheet F'':

$$
\begin{aligned}
&z'' - 2\frac{q_{21}}{p_{21}}z' - \sigma' - q_{22}z + \frac{q_{21}}{p_{21}}\sigma = 0,\\
(16)\qquad &\sigma'' + [2(q_{11}+q_{22}) - p_{12}p_{21}]z' - 2\frac{q_{21}}{p_{21}}\sigma'\\
&\quad + \left[2q_{22}' - p_{21}q_{12} - 4\frac{q_{21}}{p_{21}}q_{22}\right]z - q_{11}\sigma = 0.
\end{aligned}
$$

In these equations C_y and C_z are the two branches of the flecnode curve of S; C_ϱ and C_σ are two arbitrary curves on the two sheets of the flecnode surface. It has, moreover, been assumed that in (A), $p_{11} = p_{22} = 0$.

It may easily be verified, that if the two sheets coincide, i. e. if $\Theta_4 = 0$, the single sheet of the flecnode surface is still given by (15), if we there put $q_{22} = q_{11}$.

Examples.

Ex. 1. Find the system (A) determined by the two conics

$$
\begin{aligned}
&y_1 = 0, \quad y_2 = 1, \quad y_3 = x, \quad y_4 = x^2,\\
&z_1 = 1, \quad z_2 = x, \quad z_3 = x^2, \quad z_4 = x.
\end{aligned}
$$

Determine its flecnode curve and flecnode surface.

Ex. 2. Solve the same problem for the ruled surface determined by the curves

$$
\begin{aligned}
&y_1 = 0, \quad y_2 = 1, \quad y_3 = x, \quad y_4 = x^2,\\
&z_1 = 1, \quad z_2 = x, \quad z_3 = x^2, \quad z_4 = x^3.
\end{aligned}
$$

Ex. 3. Find the asymptotic curves, flecnode curve and flecnode surface of the integrating ruled surface of

$$y'' = ay + bz, \quad z'' = cy + dz,$$

where $a, \ldots d$ are constants.

Ex. 4 Set up the differential equations for the various classes of ruled surfaces of the fourth order (cf. for example *Jessop*'s Treatise on the Line Complex, Chapter V). Determine their flecnode curves and surfaces. Also their asymptotic curves.

CHAPTER VII.

ELEMENTS OF LINE GEOMETRY.

§ 1. Line coordinates, complexes, congruences, ruled surfaces.

A point is determined by three coordinates, a fact, which may be expressed in the language of *Lie* by saying that there are ∞^3 points in space. There are likewise ∞^3 planes in space. The older geometry considered only points as elements, all other configurations being looked upon as being composed of points. The general formulation of the principle of duality by *Poncelet* in 1822 led necessarily to the consideration of the plane as a space element, thus giving rise to a broader view of the problems of geometry. The idea that the straight line may be employed as a space-element was first formulated by *Plücker* in 1846, and has been of inestimable value for the development of geometry. Although some of the configurations of line-geometry were studied by other mathematicians previous to *Plücker*, it is from the explicit formulation of this principle that the existence of line-geometry must be dated. And it is also a consequence of the large views of *Plücker* that nowadays geometers are ready to introduce as element of space any configuration which may happen to be especially well fitted for the purpose of the problem in hand.

The first line-coordinates introduced by *Plücker* were very imperfect. If x, y, z are cartesian coordinates, a line may be (in general) represented by the two equations

$$x = rz + \varrho, \quad y = sz + \sigma,$$

where the four constants r, s, ϱ, σ are characteristic of the line. These four quantities may, therefore, be taken as the coordinates of the line. For, to every line corresponds one set of these four quantities r, s, ϱ, σ and conversely. There are, therefore, ∞^4 lines in space. These quantities are the line-coordinates introduced by *Plücker* in 1846.

If any projective transformation be made, the line is converted into another

$$x = r'z + \varrho', \quad y = s'z + \sigma'.$$

Without going into the details of the computation, we must nevertheless state the result of such a transformation. It is found that $r', s', \varrho', \sigma'$ are expressed in terms of r, s, ϱ, σ as fractions with a common denominator, this denominator and the numerators being linear functions of r, s, ϱ, σ and of

$$\eta = r\sigma - s\varrho,$$

with constant coefficients. An equation of degree n between r, s, ϱ, σ would therefore have its degree changed by a projective transformation.

This disadvantage of the above system of four coordinates was avoided by the introduction of η as a fifth, supernumerary, coordinate. This step was taken by *Plücker* in 1865. The degree of an equation between $r, s, \varrho, \sigma, \eta$ remains invariant under projective transformation, and may therefore serve as a characteristic for a configuration of line-geometry in the same way as the degree or class of a surface in point- or plane-geometry.

But these coordinates remain cumbersome, being unhomogeneous. We have already introduced homogeneous line-coordinates in Chapter II, § 6 in accordance with the general notions due to *Grassmann,* which were explained there. Essentially the same line-coordinates were employed by *Cayley*, in 1859, who showed that by means of them it becomes possible to characterize a space-curve analytically, by means of a single equation. To *Cayley*, also, is due the quadratic relation between the six homogeneous line-coordinates.

We repeat the definition. Let $y_1, \ldots y_4$ and $z_1, \ldots z_4$ be two points of the line. Put

(1) $$\omega_{ik} = y_i z_k - y_k z_i \quad (i, k = 1, 2, 3, 4).$$

Since $\omega_{ii} = 0$, and $\omega_{ik} = -\omega_{ki}$, we need retain only six of these quantities, say

$$\omega_{12}, \ \omega_{13}, \ \omega_{14}, \ \omega_{23}, \ \omega_{42}, \ \omega_{34}.$$

We define these to be the six homogeneous coordinates of the line. The propriety of this definition has already been explained. There is a one-to-one correspondence between the lines of space and the ratios of the above six quantities. There is, of course, a relation between these six quantities, since there are not ∞^5, but only ∞^4 lines in space. This relation has already been found to be (cf. Chapter II, § 6),

(2) $$\Omega = \omega_{12}\omega_{34} + \omega_{13}\omega_{42} + \omega_{14}\omega_{23} = 0,$$

where Ω may be used as an abbreviation for the left member. Conversely any six quantities which satisfy (2) may be interpreted as homogeneous coordinates of a line.

It is easy to see that, corresponding to any projective transformation of space, the six homogeneous line-coordinates ω_{ik} undergo a homogeneous linear substitution which, of course, leaves (2) invariant.

A line may be determined as the intersection of two planes, instead of being considered as joining two points. If $u_1, \ldots u_4$ and $v_1, \ldots v_4$ are the coordinates of two planes which contain the line, the determinants

$$\tau_{ik} = u_i v_k - u_k v_i$$

may also be defined as coordinates of the line. These new coordinates τ_{ik} are defined in a fashion dual to the definition of the first

set ω_{ik}, and line-geometry is clearly a self-dual field, its element being self-dual. As a consequence of this we shall see that the quantities τ_{ik} are proportional to the quantities ω_{lm}, the indices being complementary.

In fact, let the line of intersection of the planes (u), (v), coincide with the line joining the points (y), (z). Moreover, let the plane coordinates be chosen in such a way that the relation of united position for a point $(x_1, \dots x_4)$ and a plane $(w_1, \dots w_4)$ assumes the form

$$w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = 0.$$

Then we shall have

$$\begin{aligned} u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 &= 0, \\ u_1 z_1 + u_2 z_2 + u_3 z_3 + u_4 z_4 &= 0, \\ v_1 y_1 + v_2 y_2 + v_3 y_3 + v_4 y_4 &= 0, \\ v_1 z_1 + v_2 z_2 + v_3 z_3 + v_4 z_4 &= 0. \end{aligned}$$

If we eliminate successively u_1, u_2, u_3, u_4 from the first two equations, we find

$$(3) \qquad \begin{aligned} * \quad + \omega_{12} u_2 + \omega_{13} u_3 + \omega_{14} u_4 &= 0, \\ \omega_{21} u_1 + \quad * \quad + \omega_{23} u_3 + \omega_{24} u_4 &= 0, \\ \omega_{31} u_1 + \omega_{32} u_2 + \quad * \quad + \omega_{34} u_4 &= 0, \\ \omega_{41} u_1 + \omega_{42} u_2 + \omega_{43} u_3 + \quad * \quad &= 0, \end{aligned}$$

which are the conditions satisfied by a line (ω_{ik}) which lies in a plane (u_k). The skew symmetric determinant of this system of equations is equal to Ω^2 and, therefore, vanishes. Of course there is a similar system with v_k in place of u_k. Thus we shall have

$$\begin{aligned} \omega_{12} u_2 + \omega_{13} u_3 + \omega_{14} u_4 &= 0, \\ \omega_{12} v_2 + \omega_{13} v_3 + \omega_{14} v_4 &= 0, \end{aligned}$$

whence, eliminating ω_{12},

$$\omega_{13} : \omega_{14} = \tau_{12} : \tau_{23}.$$

In the same way we find the other terms of the proportion

$$(4) \qquad \tau_{12} : \tau_{13} : \tau_{14} : \tau_{23} : \tau_{42} : \tau_{34} = \omega_{34} : \omega_{12} : \omega_{23} : \omega_{14} : \omega_{13} : \omega_{12},$$

which we were to prove.

The six quantities τ_{ik} satisfy the quadratic relation

$$(5) \qquad T = \tau_{12}\tau_{34} + \tau_{13}\tau_{42} + \tau_{14}\tau_{23} = 0$$

analogous to (2).

Let $(y_1, \dots y_4)$, $(z_1, \dots z_4)$, $y_1', \dots y_4')$, $(z_1', \dots z_4')$ be the coordinates of four points which are in the same plane. Then

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \end{vmatrix} = 0,$$

which becomes, after expansion,

(6) $$\omega_{12}\omega'_{34} + \omega'_{12}\omega_{34} + \omega_{13}\omega'_{42} + \omega'_{13}\omega_{42} + \omega_{14}\omega'_{23} + \omega'_{14}\omega_{23} = 0,$$

where

$$\omega_{ik} = y_i z_k - y_k z_i, \quad \omega'_{ik} = y_i' z_k' - y_k' z_i'.$$

Therefore (6) expresses the condition that two lines ω_{ik} and ω'_{ik} shall intersect. This condition may be written

(6 a) $$\Sigma \omega'_{ik} \frac{\partial \Omega}{\partial \omega_{ik}} = 0.$$

We have seen that the six quantities ω_{ik} determine a line if they satisfy the condition $\Omega = 0$. Let us adjoin to this relation, which is satisfied by the coordinates of all lines of space, another equation between the line coordinates. In order that this new equation may be capable of geometrical interpretation, it must be homogeneous. For, since the coordinates ω_{ik} are homogeneous, the quantities $c\omega_{ik}$ represent the same line as ω_{ik} if c is an arbitrary constant. The equation of a line locus must, therefore, remain unchanged if $c\omega_{ik}$ is put in place of ω_{ik}, i. e. it must be homogeneous. Let

(7) $$\varphi(\omega_{ik}) = 0$$

be such a homogeneous equation, distinct from $\Omega = 0$. Then it is clear that the totality of lines, whose coordinates satisfy this equation, depends upon three independent parameters or, in other words, this totality consists of ∞^3 straight lines. Such a locus of ∞^3 straight lines has been called, by *Plücker*, a *line complex.*

If the equation (7) of the complex is algebraic, of the n^{th} degree the complex is said to be of the n^{th} degree.

We may write (7) as follows.

$$\varphi(y_i z_k - y_k z_i) = 0.$$

Let us regard $y_1, \dots y_4$ as constants. The equation becomes homogeneous and of the n^{th} degree in $z_1, \dots z_4$. Moreover we know that if $(z_1, \dots z_4)$ is a point which satisfies this equation, any point of the line joining it to $(y_1, \dots y_4)$ will also satisfy it. The equation represents, therefore, a cone of the n^{th} order with its vertex at the point $(y_1 \dots y_4)$. In other words: the straight lines of a complex of the n^{th} degree which pass through a given point of space, are the generators of a cone of the n^{th} order whose vertex is the given point. By means of the complex, therefore, there corresponds to every point of space

a cone of the n^{th} order with its vertex at that point. This cone may be called the *complex cone of the point.*

In place of the coordinates ω_{ik} we may introduce into (7) the coordinates π_{ik} which are proportional to them. We shall then find an equation of the form

$$\psi(u_i v_k - u_k v_i) = 0,$$

also of the n^{th} degree. By the same reasoning as above, we find that the lines of a complex of the n^{th} degree, which are situated in a given plane, envelop a curve of the n^{th} class, the *complex curve of the plane* considered.

The locus of all lines which satisfy two independent equations, homogeneous in the line coordinates, consists of ∞^2 straight lines. It is known as a *congruence.* Clearly the lines common to two complexes form a congruence. But a congruence need not be the complete intersection of two complexes, just as a space curve need not be the complete intersection of two surfaces. The essential part of our definition of a congruence is that it contains ∞^2 straight lines. The locus of ∞^1 straight lines is a ruled surface, which may or may not be the complete intersection of three complexes.

We shall have occasion to make use of the expression of the homogeneous line-coordinates in terms of non-homogeneous point-coordinates For this purpose we need merely put

$$y_1 = x, \quad y_2 = y, \quad y_3 = z, \quad y_4 = 1,$$
$$z_1 = x', \quad z_2 = y', \quad z_3 = z', \quad z_4 = 1,$$

so that

$$\omega_{12} = xy' - x'y, \quad \omega_{13} = xz' - x'z, \quad \omega_{14} = x - x',$$
$$\omega_{23} = yz' - y'z, \quad \omega_{42} = y' - y, \quad \omega_{34} = z - z'.$$

In particular the two points x, y, z and x', y', z' may be infinitesimally close to each other, so that

$$x' = x + dx, \quad y' = y + dy, \quad z' = z + dz.$$

Then the line coordinates become

$$\begin{array}{lll} x\,dy - y\,dx, & x\,dz - z\,dx, & dx, \\ y\,dz - z\,dy, & dy, & dz, \end{array}$$

in which form they have been employed principally by *Lie.* If x is the independent variable, and $\frac{dy}{dx}$ and $\frac{dz}{dx}$ are denoted by y' and z', they become

$$xy' - y, \quad xz' - z, \quad 1, \quad yz' - zy', \quad y', \quad z',$$

in which form *Halphen* has made use of them.

§ 2. The linear complex. Null-system.

Let us consider in greater detail the case in which the equation of the complex is of the first degree. Such a complex is known as a *linear complex*. Let its equation be

$$(8)\qquad \omega_{12}a_{12}+\omega_{13}a_{13}+\omega_{14}a_{14}+\omega_{23}a_{23}+\omega_{42}a_{42}+\omega_{34}a_{34}=0.$$

There is one case, in which the interpretation of this equation is obvious. Suppose that the coefficients a_{ik} satisfy the condition

$$(9)\qquad A=a_{12}a_{34}+a_{13}a_{42}+a_{14}a_{23}=0.$$

Then a_{ik} are the coordinates of a fixed line, and (8) is the condition that the line (ω_{ik}) shall intersect the line (a_{ik}) [cf. equ. (6)]. Therefore, if condition (9) is satisfied, the linear complex consists of all of the lines which intersect a given line, its axis. The line-coordinates of the axis are proportional to the coefficients of the equation (8) of the complex, taken however with complementary indices. In this case the linear complex is said to be a *special linear complex*. The equation (8) may be regarded as the equation of a straight line in line-coordinates. In the same way any curve may be represented analytically by a single equation between line-coordinates, viz.: by the equation of the complex made up of all the ∞^3 lines which intersect the curve. It was principally this fact that lead *Cayley* to introduce line-coordinates.

Let A be different from zero. According to the general theorems of § 1, we know that the lines of the complex, which pass through any point P, form a plane pencil with its vertex at P. Let p be the plane of this pencil. Then the lines of the complex which are situated in p, intersect in P. The complex determines, therefore, an involutory one-to-one correspondence between the points and the planes of space. Moreover, the corresponding elements are in united position, i. e. P lies in p and p passes through P. p is called the *null-plane* of P; P the *null-point* of p. The correspondence itself is usually spoken of as a *null-system*.

It is easy to set up the analytical expression for this correspondence. If, in (8), we introduce the explicit expressions for ω_{ik} and put $-a_{24}$ in place of a_{42}, we shall find

$$(10)\qquad \begin{aligned}&(\;*\;-a_{12}y_2-a_{13}y_3-a_{14}y_4)z_1\\ +&(a_{12}y_1+\;*\;-a_{23}y_3-a_{24}y_4)z_2\\ +&(a_{13}y_1+a_{23}y_2+\;*\;-a_{34}y_4)z_3\\ +&(a_{14}y_1+a_{24}y_2+a_{34}y_3+\;*\;)z_4=0.\end{aligned}$$

For a fixed point $(y_1,\dots y_4)$, this is clearly the equation of the plane which corresponds to it. If $u_1,\dots u_4$ are the coordinates of this

plane, they will be proportional to the coefficients of $z_1, \dots z_4$ in the above equation.

Whenever the coordinates $u_1, \dots u_4$ of a plane are given as homogeneous linear functions of the coordinates $x_1, \dots x_4$ of a point, we have a so-called dualistic transformation, giving rise to a one-to-one correspondence between points and planes. Let

$$(11) \qquad \varrho u_k = a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + a_{k4}x_4, \quad (k = 1, 2, 3, 4),$$

where ϱ is a proportionality factor, be such a dualistic transformation, and let us assume that it has the further property that every point lies in the plane which corresponds to it Such is the case, as we have seen, in the point-plane correspondence determined by a linear complex. We must then have

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0$$

for all values of $x_1, \dots x_4$. We find immediately

$$a_{11} = a_{22} = a_{33} = a_{44} = 0, \quad a_{ki} = -a_{ik},$$

i. e. the determinant of (11) must be skew-symmetric. But this gives precisely the point-plane transformation determined by the linear complex. Therefore:

The *point-plane correspondence determined by a linear complex is the most general dualistic correspondence, for which all pairs of corresponding elements are in united position*

We have found that the coordinates $(u_1, \dots u_4)$ of the plane, which corresponds to the point $(x_1, \dots x_4)$, are given by the equations

$$(12) \qquad \begin{aligned} \varrho u_1 &= \quad * \quad - a_{12}x_2 - a_{13}x_3 - a_{14}x_4, \\ \varrho u_2 &= a_{12}x_1 + \quad * \quad - a_{23}x_3 - a_{24}x_4, \\ \varrho u_3 &= a_{13}x_1 + a_{23}x_2 + \quad * \quad - a_{34}x_4, \\ \varrho u_4 &= a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + \quad * \quad . \end{aligned}$$

Put

$$x_i = y_i + \lambda z_i,$$

i. e. let the point P_x describe the line P_yP_z. Then we shall find

$$\varrho u_i = v_i + \lambda w_i,$$

where $(v_1, \dots v_4)$ and $(w_1, \dots w_4)$ are the coordinates of the planes which correspond to P_y and P_z respectively; i. e as a *point P_x describes a straight line, the corresponding plane turns about another line as axis.* This relation between the two lines is reciprocal. They are said to be *reciprocal polars* of each other with respect to the complex. It is easy to show further, that *every line of the complex is reciprocal polar to itself, and that every line, which intersects two reciprocal polars, belongs to the complex.*

The determinant of (12) is equal to

$$(13)\qquad \begin{vmatrix} 0, & -a_{12}, & -a_{13}, & -a_{14} \\ a_{12}, & 0, & -a_{23}, & -a_{24} \\ a_{13}, & a_{23}, & 0, & -a_{34} \\ a_{14}, & a_{24}, & a_{34}, & 0 \end{vmatrix} = (a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23})^2 = A^2.$$

The equation of the complex may be written

$$(14)\qquad \sum_{\iota, k=1}^{4} a_{\iota k}\omega_{\iota k} = 0,$$

if we put $a_{\iota\iota} = 0$, $a_{\iota k} = -a_{k\iota}$. A general projective transformation

$$y_\iota = \sum_{k=1}^{4} b_{\iota k} y_k', \quad z_\iota = \sum_{k=1}^{4} b_{\iota k} z_k',$$

transforms the line coordinates $\omega_{\iota k}$ in accordance with the equations

$$(15)\qquad \omega_{\iota k} = \sum_{\mu, \nu=1}^{4} b_{\iota\mu} b_{k\nu} \omega'_{\mu\nu}, \quad (i, k = 1, 2, 3, 4)$$

The result of this transformation upon the complex (14), will be to convert it into

$$\sum_{\iota, k, \mu, \nu=1}^{4} a_{\iota k} b_{\iota\mu} b_{k\nu} \omega'_{\mu\nu} = 0,$$

i. e. into another linear complex whose coefficients $a'_{\mu\nu}$ are given by the equations:

$$(16)\qquad a'_{\mu\nu} = \sum_{\iota, k=1}^{4} a_{\iota k} b_{\iota\mu} b_{k\nu}, \quad (\mu, \nu = 1, 2\ 3, 4).$$

If we denote the quantity A for this new linear complex by A', A'^2 may be written in the form of a skew-symmetric determinant corresponding to (13). If we denote the determinant of the quantities $b_{\iota k}$ by $\varDelta$, and if we make use of equations (16), together with the rule for the multiplication of determinants, we shall find

$$A'^2 = \varDelta^2 A^2,$$

i. e. A is a relative invariant of the complex for projective transformations.

That a linear complex has no absolute invariant under projective transformation may be seen as follows. Let us choose the tetrahedron of reference in such a way, that two of its non-intersecting edges, say $x_1 = x_2 = 0$ and $x_3 = x_4 = 0$ are reciprocal polars with respect to the complex. Then the plane, which corresponds to any point of the edge $x_1 = x_2 = 0$, must contain the other edge; i. e. for $x_1 = x_2 = 0$ we must find $u_1 = u_2 = 0$, for all values of x_3 and x_4. But equations

(12) show that this is possible only if $a_{13} = a_{14} = a_{23} = a_{34} = 0$. The equation of the complex reduces to

$$a_{12}\omega_{12} + a_{34}\omega_{34} = 0$$

or

(17) $$\omega_{12} - k\omega_{34} = 0,$$

the invariant of which is

(18) $$A = -k.$$

If $A = 0$ this becomes

(19) $$\omega_{12} = 0.$$

If $A \neq 0$, the transformation

$$x_1 = k\bar{x}_1, \quad x_2 = \bar{x}_2, \quad x_3 = \bar{x}_3, \quad x_4 = \bar{x}_4,$$

converts the equation into

(20) $$\omega_{12} - \omega_{34} = 0.$$

Therefore, by a projective transformation every special linear complex can be converted into (19), and every non-special linear complex into (20).

In other words, every special linear complex can be transformed projectively into any other, and every non-special linear complex into any other non-special linear complex. *The linear complex, therefore, has no absolute invariant under projective transformations.*

Since the equation of a linear complex depends upon the five ratios of the coefficients a_{ik}, it is clear that *a linear complex is determined, in general, by five of its lines.* The exceptional case, when five lines do not determine a linear complex, will be easily understood after the developments of the next paragraph. The complex is also determined by a pair of reciprocal polars p, p' and one of its lines l which does not intersect p and p'. For any four lines, which intersect p and p', are also lines of the complex. It is also determined by two pairs of reciprocal polars. But not both of these pairs can be assigned arbitrarily.

For other properties of the linear complex, especially for a complete discussion of the arrangement of its lines, the reader may consult: *Plücker's Neue Geometrie des Raumes; Clebsch-Lindemann, Vorlesungen über Geometrie, vol. II; Jessop, A treatise on the Line complex.*

§ 3. The linear congruence.

The lines common to two linear complexes

(21) $$\begin{aligned} A &= a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{42}\omega_{42} + a_{34}\omega_{34} = 0, \\ M &= b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{42}\omega_{42} + b_{34}\omega_{34} = 0, \end{aligned}$$

form a linear congruence. They belong also to each of the ∞^1 complexes

$$\lambda \mathit{\Lambda} + \mu M = 0. \tag{22}$$

If we denote the invariants of $\mathit{\Lambda} = 0$ and $M = 0$ by A and B respectively, and if we put

$$C = a_{12} b_{34} + a_{23} b_{14} + a_{31} b_{42} + a_{34} b_{12} + a_{14} b_{23} + a_{42} b_{31}, \tag{23}$$

the invariant of (22) will be

$$\lambda^2 A + \lambda \mu C + \mu^2 B.$$

If, therefore,

$$C^2 - 4AB \neq 0,$$

there will be two linear complexes of the family (22) which are special. In other words, *the lines of the congruence are the common intersectors of two straight lines, the directrices of the congruence.*

The directrices of the congruence will be skew to each other. For, suppose that they were coplanar. Then *every* line of their plane would belong to *all* of the complexes (22). But, unless a linear complex is special, all of the lines of any plane which belong to it, form a plane pencil. Therefore, under our supposition, all of the complexes (22) would be special complexes, i. e. we would have

$$A = B = C = 0$$

In this case, the congruence degenerates into two systems of ∞^2 lines, viz.: the lines of the plane of the two directrices, and the lines through their point of intersection.

If

$$C^2 - 4AB = 0,$$

we may speak of a congruence with coincident directrices. Such a congruence will be obtained as a limiting case of a congruence with distinct directrices if we allow these directrices to approach each other as a limit without, however, becoming coplanar. This way of looking at such a congruence shows that its lines may be regarded as the tangents of a hyperboloid along one of its generators. For, these tangents intersect two consecutive generators of the same set upon the hyperboloid. The lines of the congruence may, therefore, be arranged in a single infinity of plane pencils. The vertices of all of these pencils lie upon the directrix. As the vertex of the pencil moves along the directrix, its plane turns around the directrix as axis; the point-row and the pencil of planes thus generated are projective to each other. In fact they stand in the same relation to each other as the points of the generator of a hyperboloid to their tangent planes.

It is clear now that five lines determine a linear complex, provided that they do not belong to a linear congruence. It is also evident that four lines, in general, determine a linear congruence, its

directrices being the two straight line intersectors of the given four lines. There is, of course, an exception if the given four lines are generators of the same hyperboloid, or if they intersect.

We give, without proof, a few other theorems which we shall employ occasionally.

Let $\Lambda = 0$, $M = 0$ be two linear complexes, and consider any four complexes

$$\Lambda + k_i M = 0 \quad (i = 1, 2, 3, 4)$$

of the pencil of complexes determined by them. To any point of space there corresponds, in each of these complexes, a plane. These four planes form a pencil whose double ratio is equal to (k_1, k_2, k_3, k_4). To any plane there corresponds, in each of these complexes, a point. These four points are collinear, and their double-ratio is equal to (k_1, k_2, k_3, k_4).

We may, therefore, speak of the double-ratio of the four complexes

$$\Lambda + k_i M = 0,$$

defining it to be equal to (k_1, k_2, k_3, k_4).

Let two of these four complexes be the special complexes of the pencil (supposed distinct). Let the other two be chosen in such a way that the double-ratio of the four complexes becomes equal to -1. The two latter complexes are then said to be in *involution*. It is not difficult to show that the condition for two linear complexes in involution is

$$C = 0.$$

In the cases in which our former definition breaks down, the equation $C = 0$ may be taken as the definition of the involutory relation between two linear complexes.[1])

Finally, we may call attention to the fact that *Lie* has set up a geometry whose element is the sphere. This geometry is four-dimensional as is *Plücker*'s line-geometry. By making use of a simple transformation due to *Lie*, the two geometries may be converted into each other, a line in one corresponding to a sphere in the other. It is a mere matter of convenience in most cases, whether a given analytical theorem is to be interpreted in line- or in sphere-geometry. In place of ruled surfaces we would have surfaces generated by moving spheres, in place of asymptotic lines, lines of curvature, etc. We shall not enter into details, but leave it to the reader to re-interpret the theorems about ruled surfaces in sphere-geometry.[2])

1) This idea of two linear complexes in involution is due to *Klein* Math. Ann. vol. 2 (1870) p. 198. For a proof of the above theorems cf. *Lie-Scheffers*, Geometrie der Berührungstransformationen, pp. 296—300.

2) For a convenient treatment of this subject, cf. *Lie-Scheffers*, Geometrie der Berührungstransformationen, pp. 453 et sequ.

CHAPTER VIII.

THE EQUATION OF THE RULED SURFACE IN LINE COORDINATES.

§ 1. The differential equation for the line coordinates.

We return to the consideration of a system of differential equations of the form (A), and put

$$\omega_{ik} = y_i z_k - y_k z_i, \tag{1}$$

so that ω_{ik} will be the Plückerian coordinates of the generator L_{yz} of the integrating ruled surface. Of course the identical relation

$$\omega_{12}\omega_{34} + \omega_{13}\omega_{42} + \omega_{23}\omega_{14} = 0 \tag{2}$$

will hold.

Corresponding to the infinite group G composed of all of the transformations

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z, \quad \bar{x} = \xi(x), \tag{3}$$

the functions ω_{ik} will be transformed in accordance with the equations

$$\bar{\omega} = (\alpha\delta - \beta\gamma)\,\omega = \varphi\,\omega, \quad \bar{x} = \xi(x). \tag{4}$$

Now the six line coordinates ω_{ik} will satisfy a linear homogeneous differential equation of the sixth order, say

$$P_0\omega^{(6)} + P_1\omega^{(5)} + P_2\omega^{(4)} + P_3\omega^{(3)} + P_4\omega'' + P_5\omega' + P_6\omega = 0, \tag{5}$$

where $P_0, \ldots P_6$ are functions of the coefficients p_{ik}, q_{ik} of (A).

Clearly, *the invariants of the linear differential equation of the sixth order, which the line coordinates of a generator of the integrating ruled surface of system* (A) *satisfy, will also be invariants of system* (A), *and conversely.*

We have found in Chapter II, a system of rational invariants for (5), complete in the sense that every other rational invariant can be expressed rationally in terms of them. *We may, therefore, consider the problem of finding such a system of invariants for system* (A) *as essentially solved.*

We proceed to set up the differential equation (5) in a special form. Let us assume that the system (A) is written in the semi-canonical form, and let (y, z) and (η, ζ) be any two simultaneous systems of solutions of this system, so that

$$\begin{aligned} y'' &= -q_{11}y - q_{12}z, & \eta'' &= -q_{11}\eta - q_{12}\zeta, \\ z'' &= -q_{21}y - q_{22}z, & \zeta'' &= -q_{21}\eta - q_{22}\zeta. \end{aligned} \tag{6}$$

Then put

$$\omega = y\xi - z\eta = (y\xi). \tag{7}$$

We wish to find the differential equation satisfied by ω. We find from (6),

$$(\eta y'') = +q_{12}\,\omega, \quad (\eta z'') = +q_{22}\,\omega, \quad (\xi y'') = -q_{11}\,\omega, \quad (\xi z'') = -q_{21}\,\omega, \tag{8}$$

where we have put for abbreviation

$$\begin{aligned} (\eta y'') &= \eta y'' - y\eta'', \quad (\eta z'') = \eta z'' - y\xi'', \\ (\xi y'') &= \xi y'' - z\eta'', \quad (\xi z'') = \xi z'' - z\xi''. \end{aligned}$$

In general we have denoted by $(a\beta)$ the expression

$$a\beta - \alpha b,$$

obtained from the term actually written by substracting a corresponding term, in which the Greek and Roman letters are interchanged.

Put

$$\begin{aligned} 2v &= \omega'' + (q_{11} + q_{22})\,\omega, \\ w &= v'' - 2\,(q_{11}q_{22} - q_{12}q_{21})\,\omega + (q_{11} + q_{22})\,v, \\ t &= w' - (q_{11}q'_{22} + q_{22}q'_{11} - q_{12}q'_{21} - q_{21}q'_{12})\,\omega + (q'_{11} + q'_{22})\,v, \\ \varrho &= t' - (q_{11}q''_{22} + q_{22}q''_{11} - q_{12}q''_{21} - q_{21}q''_{12})\,\omega + (q''_{11} + q''_{22})\,v. \end{aligned} \tag{9}$$

Then we shall find from (7), by successive differentiation and by making use of (8) and (9),

$$\begin{aligned} \omega' &= -(\eta z') + (\xi y'), \\ v' &= q_{11}(\eta z') + q_{12}(\xi z') - q_{21}(\eta y') - q_{22}(\xi y'), \\ w &= q'_{11}(\eta z') + q'_{12}(\xi z') - q'_{21}(\eta y') - q'_{22}(\xi y'), \\ t &= q''_{11}(\eta z') + q''_{12}(\xi z') - q''_{21}(\eta y') - q''_{22}(\xi y'), \\ \varrho &= q^{(3)}_{11}(\eta z') + q^{(3)}_{12}(\xi z') - q^{(3)}_{21}(\eta y') - q^{(3)}_{22}(\xi y'). \end{aligned} \tag{10}$$

If we eliminate the four determinants $(\eta z')$, etc., we find the required differential equation of the sixth order for ω, viz.:

$$\begin{vmatrix} \varrho, & q^{(3)}_{11}, & q^{(3)}_{12}, & -q^{(3)}_{21}, & -q^{(3)}_{22} \\ t, & q''_{11}, & q''_{12}, & -q''_{21}, & -q''_{22} \\ w, & q'_{11}, & q'_{12}, & -q'_{21}, & -q'_{22} \\ v', & q_{11}, & q_{12}, & -q_{21}, & -q_{22} \\ \omega', & -1, & 0, & 0, & +1 \end{vmatrix} = 0, \tag{11}$$

where ϱ, t, w, v' are defined by equations (9).

§ 2. Conditions for a ruled surface whose generators belong to a linear complex or a linear congruence.

Equation (11) is, in general, of the sixth order, so that the six line coordinates will be linearly independent. It may, however, reduce to the fifth order. In that case there must be a linear homogeneous relation with constant coefficients between the line-coordinates, i. e. the ruled surface must belong to a linear complex.

If we recur to equations (9), we see that the only one of the quantities ϱ, t, w, v', which contains $\omega^{(6)}$, is ϱ. The equation (11) will, therefore, reduce to the fifth order, if and only if the minor of ϱ in (11) is zero, i. e. if

$$(12) \qquad \begin{vmatrix} q''_{11} - q''_{22}, & q''_{12}, & q''_{21} \\ q'_{11} - q'_{22}, & q'_{12}, & q'_{21} \\ q_{11} - q_{22}, & q_{12}, & q_{21} \end{vmatrix} = 0.$$

But the invariant $\varDelta$ of system (A) reduces to the left member of (12), except for a numerical factor, if (A) is reduced to its semi-canonical form.

Therefore, *the condition that a ruled surface may belong to a linear complex is*

$$(13) \qquad \Theta_9 = \varDelta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{12}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix} = 0.$$

If the linear complex, to which the generators of the surface belong, is special, additional conditions must be fulfilled. The surface has, in that case, a straight line directrix. This straight line directrix is clearly both a branch of the flecnode curve and an asymptotic curve upon the surface, so that we shall have

$$u_{12} = p_{12} = 0$$

if the directrix be taken as fundamental curve C_y. If the surface does not belong to a second linear complex, the invariant Θ_4 must be different from zero. For else the second branch of the flecnode curve would also be a straight line, coincident with the first, and the surface would belong to a linear congruence with coincident directrices. We shall, therefore, have

$$u_{12} = p_{12} = 0, \quad u_{11} - u_{22} \neq 0, \quad q_{12} = 0,$$

whence follows $v_{12} = 0$, and therefore

$$(14) \qquad \Theta_4 \neq 0, \quad \Theta_{10} = 0.$$

The conditions (14) *are therefore necessary for a ruled surface with a straight line directrix. They are also sufficient.* For, let them be

satisfied. Then the two branches of the flecnode curve are distinct, and if we identify them with the integral curves C_y and C_z of system (A), we shall have

$$u_{12} = u_{21} = 0, \quad u_{11} - u_{22} \neq 0,$$

so that

$$\Theta_{10} = -(u_{11} - u_{22})^2 v_{12} v_{21}.$$

If $\Theta_{10} = 0$, we must, therefore, have either v_{12} or v_{21} equal to zero. But

$$(15) \qquad \begin{aligned} v_{12} &= 2u'_{12} + (p_{11} - p_{22}) u_{12} - p_{12}(u_{11} - u_{22}), \\ v_{21} &= 2u'_{21} - (p_{11} - p_{22}) u_{21} + p_{21}(u_{11} - u_{22}). \end{aligned}$$

We find, therefore, either p_{12} or p_{21} equal to zero, i. e. either C_y or C_z is a straight line.

Of course Δ also vanishes in consequence of these conditions.

If the ruled surface belongs to a linear congruence with distinct directrices, it has two distinct straight line directrices upon it, so that system (A) may be reduced to a form for which

$$p_{12} = u_{12} = 0, \quad p_{21} = u_{21} = 0, \quad u_{11} - u_{22} \neq 0.$$

In this case we shall find further

$$v_{12} = v_{21} = w_{12} = w_{21} = 0,$$

so that all of the minors of the second order in the determinant Δ will vanish, as well as Θ_{10}. On the other hand, suppose that all of the minors of the second order in Δ are zero, while Θ_4 does not vanish. Identify C_y and C_z with the (distinct) branches of the flecnode curve. We shall have

$$u_{12} = u_{21} = 0, \quad u_{11} - u_{22} \neq 0$$

Two of the minors of Δ reduce to

$$(u_{11} - u_{22}) v_{12} \quad \text{and} \quad (u_{11} - u_{22}) v_{21},$$

so that we must have

$$v_{12} = v_{21} = 0.$$

But, from (15), we now find $p_{12} = p_{21} = 0$, i. e. C_y and C_z are two distinct straight lines

Therefore, *the conditions, necessary and sufficient for a ruled surface with two distinct straight line directrices, are that all of the minors of the second order in the determinant Δ shall vanish, while $\Theta_4 \neq 0$.*

This result may be obtained in another way. The equation (12) shows, that if a ruled surface belongs to a linear complex, and if the corresponding system (A) is written in the semi-canonical form, there is a linear relation with constant coefficients

$$a(q_{11} - q_{22}) + bq_{12} + cq_{21} = 0,$$

between the coefficients of the system; and it is for this reason that the first four equations of (10) suffice for the elimination of the four determinants $(\eta z')$, etc. But, if there is a second linear relation of this kind, the first three equations of (10) will suffice for this elimination, so that the differential equation for ω will reduce to the fourth order; i. e. the ruled surface will belong to a linear congruence. On the other hand the conditions for two such linear relations between $q_{11} - q_{22}$, q_{12}, q_{21} are precisely these, that the minors of the second order in Δ shall all vanish.

If the ruled surface belongs to a linear congruence with coincident directrices, the minors of Δ must again vanish, as the last consideration shows. But Θ_4 must also be zero.

We may show this directly. Of course Θ_4 must be zero, since the two coincident directrices of the congruence are identical with the two branches of the flecnode curve, which must therefore coincide Let C_y be this straight line, so that

$$p_{12} = u_{12} = 0.$$

Let C_z be any other curve of the ruled surface.

Since Θ_4 vanishes, we shall have

$$u_{11} - u_{22} = 0.$$

We find as consequences of these relations

$$v_{11} - v_{22} = 0, \quad v_{12} = 0, \quad w_{11} - w_{22} = 0, \quad w_{12} = 0,$$

so that, in fact, all of the minors of the second order in Δ must vanish. The quantity u_{21} will not also be zero unless the surface is a quadric

Conversely let us suppose that all of these minors vanish, and that Θ_4 also is equal to zero. Take for C_y the flecnode curve, so that

$$u_{12} = u_{11} - u_{22} = 0, \quad u_{21} \neq 0$$

assuming that the surface is not a quadric. One of the minors of Δ reduces to

$$(v_{11} - v_{22})u_{21} = 2p_{12}u_{21}^2,$$

so that $p_{12} = 0$; i. e. C_y is a straight line. Moreover both branches of the flecnode curve coincide with it. It must therefore be a double directrix of the ruled surface.

The further case that presents itself in the theory of linear congruences, in which its directrices are coplanar, has no interest for us. For the lines of such a congruence are either all of the lines of a plane, or all of the lines through a point. A ruled surface belonging to such a congruence would therefore be either a cone, or

else its generators would envelop a plane curve. In either case the ruled surface would be developable; but a developable cannot be the integrating ruled surface of a system of form (A).

If a ruled surface belongs to three independent linear complexes it is a quadric. We know already that the conditions for this are

$$u_{11} - u_{22} = u_{12} = u_{21} = 0.$$

In this, as in the preceding case, all of the invariants are zero.

We may recapitulate the results of this paragraph in the following theorem.

The necessary and sufficient conditions for a ruled surface belonging to a single linear complex, which is not special, are

$$\Theta_1 \neq 0, \quad \Delta = 0, \quad \Theta_{10} \neq 0,$$

while all of the minors of the second order in Δ do not vanish. If

$$\Theta_{10} = 0,$$

while the other conditions remain the same, the complex is special. The surface belongs to a linear congruence with distinct directrices if all of the minors of the second order in Δ vanish, while Θ_4 is different from zero. The directrices of the congruence coincide if Θ_4 also vanishes. In this latter case the surface is, or is not a quadric according as the equations

$$u_{11} - u_{22} = u_{12} = u_{21} = 0$$

are, or are not satisfied.

Suppose $\Delta = 0$ and $\Theta_4 = 0$. We may put $u_{12} = u_{11} - u_{22} = 0$, and assume $u_{21} \neq 0$ if the surface is not a quadric We find from $\Delta = 0$,

$$(v_{11} - v_{22}) w_{12} - (w_{11} - w_{22}) v_{12} = 0,$$

which equation reduces to

$$- 4 p_{12}^3 u_{21}^2 = 0,$$

so that p_{12} must vanish, and C_y be, therefore, a straight line. We have the following theorem, due to *Voss.*[1])

If the two branches of the flecnode curve of a ruled surface belonging to a linear complex coincide, it is a straight line.

This gives a simpler test than that given above for a ruled surface belonging to a linear congruence with coincident directrices.

We found in Chapter V, § 3 that the identically self-dual surfaces were those for which $\Theta_9 = 0$. We may, therefore, express this result by saying: *a ruled surface, with two distinct branches to its flecnode curve, is identically self-dual, if and only if it belongs to a non-special linear complex.*

1) *Voss.* Mathematische Annalen, Bd. VIII p. 92

The proof of this theorem however is not quite complete, since we had assumed not only $\Theta_4 \neq 0$, but also $\Theta_{10} \neq 0$. We shall now consider these exceptional cases. We shall see at the same time that the fundamental theorem, that a ruled surface is determined uniquely by means of its invariants, actually ceases to be true in these cases.

First, let $\Theta_4 \neq 0$, $\Theta_{10} = 0$. The ruled surface S has a single straight line directrix, which we may identify with C_y. We shall then have

$$p_{12} = u_{12} = 0, \quad \text{whence} \quad q_{12} = 0.$$

Let C_z be the second branch of the flecnode curve of S distinct from C_y. Then

$$u_{21} = 0.$$

But, multiplying y and z by properly chosen functions of x, we may further make

$$p_{11} = p_{22} = 0,$$

so that the system (A) assumes the form:

$$(16) \qquad \begin{aligned} &y'' + q_{11}\, y = 0, \\ &z'' + p_{21}\, y' + \tfrac{1}{2} p'_{21}\, y + q_{22}\, z = 0. \end{aligned}$$

The non-vanishing invariants Θ_4, $\Theta_{4\,1}$ and Θ_6 being given as functions of x do not determine p_{21}, which may still be chosen as an arbitrary function of x. This arbitrariness does not disappear even if the independent variable of the system, which is still capable of arbitrary transformation, be chosen in a determined fashion. We may determine it so that, without disturbing the other conditions,

$$u_{11} - u_{22} = 1.$$

We shall then have

$$q_{11} - q_{22} = -\frac{1}{4},$$

but p_{21} remains an arbitrary function. Now, the most general transformation, which leaves all of these conditions invariant, is

$$\xi = \pm x + \text{const.}, \quad y = ay, \quad z = bz,$$

where a and b are constants. Evidently such a transformation cannot remove the arbitrary function p_{21}. Therefore, *if* $\Theta_4 \neq 0$, $\Theta_{10} = 0$, *the invariants do not suffice to characterize the ruled surface.*

The adjoint system of (16) is

$$(17) \qquad \begin{aligned} &U'' + q_{22}\, U = 0, \\ &V'' + p_{21}\, U' + \tfrac{1}{2} p'_{21}\, U + q_{11}\, V = 0. \end{aligned}$$

The independent variable is the same for both systems. Moreover both systems are referred to their flecnode curves. Therefore, the

only transformations, which could transform (16) into (17), must be of the form either

$$y = \alpha U, \quad z = \delta V,$$

or

$$y = \alpha V, \quad z = \delta U.$$

Moreover α and δ must both be constants, so as to preserve the conditions $p_{11} = p_{22} = 0$ which are satisfied in both systems. Since $q_{22} \neq q_{11}$, the first transformation cannot accomplish this. The second can, if and only if $p_{21} = 0$, i. e. if C_z also is a straight line. The ruled surface has two distinct straight line directrices, and therefore belongs to an infinity of non-special linear complexes. This completes the proof of our theorem about identically self-dual surfaces, if $\Theta_4 \neq 0$.

If $\Theta_4 = 0$, we must have, in the case of an identically self-dual surface $\Theta_9 = 0$ as in the general case. But this gives either a ruled surface belonging to a linear congruence with coincident directrices or else a quadric. In all of these cases the surface belongs to an infinity of non-special linear complexes.

Therefore, *a ruled surface is identically self-dual, if and only if it belongs to at least one non-special linear complex.* The dualistic transformation which converts it into itself, generator for generator, is that which consists in replacing every point of space by the plane which corresponds to it in the complex.

The invariants do not determine the surface, if $\Theta_1 = 0$. We may assume in this case

$$u_{12} = u_{11} - u_{22} = p_{11} - p_{22} = 0,$$

so that C_y is the flecnode curve. We may assume further

$$p_{21} = 0,$$

so that C_z is an asymptotic curve, and the independent variable may be chosen so as to make

$$u_{11} + u_{22} = 0.[1]$$

We find for the coefficients of a system (A) satisfying these conditions

$$\begin{array}{lllll} & p_{11} = 0, & p_{12} = f(x), & q_{11} = 0, & q_{12} = \frac{1}{2} p'_{12}, \\ (18) & p_{21} = 0, & p_{22} = 0, & q_{21} - \frac{1}{4} u_{21} = g(x), & q_{22} = 0, \\ & & p_{12} u_{21} = \frac{9\Theta_9}{4\Theta_6}, & & \end{array}$$

if $\Theta_6 \neq 0$. Of the two functions f and g one remains arbitrary. Moreover, the most general transformation, which leaves the above system of conditions invariant, contains only arbitrary constants, and cannot, therefore, remove this arbitrary function.

1) Equation (51) of Chapter IV shows how this may be done.

If Θ_6 also vanishes, we find either $u_{21} = 0$, i. e. S is a quadric, or $p_{12} = 0$, so that S belongs to a linear congruence with coincident directrices. In either case all of the invariants vanish.

We may, therefore, complete our fundamental theorem negatively as follows.

The ruled surfaces, for which the two branches of the flecnode curve coincide, and those which have a straight line directrix, form an exception to the fundamental theorem which states that a ruled surface is determined by its invariants, up to a projective transformation.

§ 3. A function-theoretic application.

Consider a homogeneous linear differential equation of the n^{th} order

(19) $$D(y) = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0,$$

and let $y_1, \ldots y_n$ form a fundamental system, so that

(20) $$D(y_i) = 0 \quad (i = 1, 2, \ldots n).$$

Let the coefficients $p_1, \ldots p_n$ of (19) be uniform functions of x, and let $x = a_\mu$ be a singular point of one or all of the coefficients. If the complex variable x describes a closed path around one of these points a_μ, $y_1, \ldots y_n$ will, in general undergo a linear substitution with constant coefficients, changing into

$$y_k = \sum_{i=1}^{n} \lambda_{ki}^{(\mu)} y_i \quad (k = 1, 2, \ldots n),$$

where the determinant $|\lambda_{ki}^{(\mu)}|$ is not equal to zero[1]) Denote this substitution by A_μ, so that we may write

(21) $$\bar{y}_k = A_\mu y_k.$$

Now, put in (20),

(22) $$y_k = \sum_{i=1}^{n} \alpha_{ki}(x) \eta_i = S \eta_k, \quad (k = 1, 2, \ldots n),$$

where again the determinant $|\alpha_{ik}|$ is different from zero, and where $\alpha_{ki}(x)$ are uniform functions of x. Then $\eta_1, \ldots \eta_n$ will satisfy a system of n linear differential equations of the n^{th} order, obtained from (20) by the transformation (22).

This system of differential equations has the special property that, when x describes a closed path around a_μ, $\eta_1, \ldots \eta_n$ undergo the linear substitution

$$S^{-1} A_\mu S,$$

whose coefficients are uniform functions of x. $\eta_1, \ldots \eta_n$ are, therefore, a special case of what the author has called A functions. In

1) *Fuchs.* Crelles Journal, vol. 66

fact, a system of functions $z_1, \dots z_n$ is known as a $\varDelta$ function if $z_1, \dots z_n$ are uniform, finite and continuous for all values of the complex variable x, except for certain values, say $a_1, a_2, \dots a_m$; and if, when x describes a closed path around one of these points $z_1, \dots z_n$ undergo a linear substitution, whose coefficients are uniform functions of x.[1])

In the case $n = 2$ we can now write down the conditions under which a system of differential equations of form (A) will have this property. There must exist a transformation

$$y = \alpha(x)\eta + \beta(x)\zeta, \quad z = \gamma(x)\eta + \delta(x)\zeta,$$

which converts the system into

$$(23) \qquad \begin{aligned} \eta'' + p\eta' + q\eta &= 0, \\ \zeta'' + p\zeta' + q\zeta &= 0, \end{aligned}$$

so that η and ζ satisfy the same linear differential equation of the second order. But for this latter system we find at once

$$(24) \qquad u_{11} - u_{22} = u_{12} = u_{21} = 0,$$

an invariant system of equations, which must therefore hold of the original system (A) as well. The integrating ruled surface must, therefore, be a quadric. The curves C_η and C_ζ are clearly any two generators of the second kind It is evident that the conditions (24) are also *sufficient* for a system (A) of the required kind, if we do not insist upon the condition that the coefficients of the substitution S shall be *uniform* functions of x, and still speak of the functions y and z as $\varDelta$ functions.

We may, therefore, say that a system of form (A) gives rise to a binary system of $\varDelta$ functions, if and only if its integrating ruled surface is a quadric.

Examples.

Ex. 1. If a ruled surface belongs to a linear complex, its differential equations may be put into such a form that

$$p_{11} = p_{22} = 0, \quad u_{12} = u_{21} = 0, \quad \frac{p_{12}}{p_{21}} = \text{const},$$

provided that Θ_4 and Θ_{10} do not vanish.

Ex. 2. The points of the straight line $y_1 = 1$, $y_2 = x$, $y_3 = y_4 = 0$, are joined to those of the conic

$$z_k = a_{k0} + a_{k1}x + a_{k2}x^2, \quad (k = 1, 2, 3, 4).$$

What are the conditions under which the ruled surface thus generated has a second straight line directrix?

1) *Wilczynski.* Am. Jour. of Math. Vol. 21, pp. 85—106 (1899).

CHAPTER IX.

THE FLECNODE CONGRUENCE.

§ 1. The developables of the congruence and its focal surface.

In Chapter VI, the covariant P led to the consideration of the hyperboloid H, which osculates a given ruled surface S along one of its generators g. It was found that the generator h of the ruled surface S', the derivative of S with respect to x, was a generator of the first kind upon H. There is an osculating hyperboloid H for each generator g of S; and upon each of these hyperboloids there is a single infinity of generators of the first kind. The totality of these generators consists, therefore, of ∞^2 straight lines; we shall speak of this congruence, composed of all of the generators of the first kind on the osculating hyperboloids of S, as the *flecnode congruence of S*, and denote it by the letter Γ. The reason for choosing this name for the congruence, will appear in the course of the present chapter.

The ruled surface S', the derivative of S with respect to x, is always a surface of the flecnode congruence, one of its generators being situated upon each of the osculating hyperboloids of S. Clearly, unless all of the osculating hyperboloids coincide, i. e. unless S is itself a quadric, the congruence Γ does not degenerate into a single infinity of lines. A transformation $\xi = \xi(x)$ of the independent variable transforms the surface S' into another ruled surface of the congruence Γ, since it transforms ϱ and σ into

$$(1) \qquad \bar{\varrho} = \frac{1}{\xi'}(\varrho + \eta y), \quad \bar{\sigma} = \frac{1}{\xi'}(\sigma + \eta z), \quad \eta = \frac{\xi''}{\xi'}.$$

Since the coordinates are homogeneous, the factor ξ' is of no importance, and two transformations, which give the same value to η, give rise to the same ruled surface of Γ. In other words, a linear transformation

$$\xi = ax + b,$$

where a and b are constants, does not affect the derivative ruled surface. On the other hand, η being an arbitrary function of x, any surface of the congruence Γ, which has one generator on each osculating hyperboloid of S, may be regarded as the derivative of S with respect to an appropriately chosen independent variable, excepting only S itself, for which η would have to be equal to infinity.

Equations (8) of Chapter VI reduce to the form

$$\alpha\varrho' + \beta\sigma' + \gamma\varrho + \delta\sigma = 0,$$

if $J = 0$. In that case, therefore, the derivative surface S' is developable [cf. Chapter V, equations (14)]. This conclusion would seem to be doubtful if, besides $J = 0$, the conditions

$$\alpha = \beta = \gamma = \delta = 0$$

were satisfied. We may, however, see directly that the condition $J=0$ always gives rise to a developable surface for S'. In fact, we have

$$(2) \qquad \begin{aligned} 2\varrho' + p_{11}\varrho + p_{12}\sigma &= u_{11}y + u_{12}z, \\ 2\sigma' + p_{21}\varrho + p_{22}\sigma &= u_{21}y + u_{22}z, \end{aligned} \qquad J = u_{11}u_{22} - u_{12}u_{21}.$$

If $\Theta_4 \neq 0$, the two branches of the flecnode curve are distinct. If we identify them with C_y and C_z, we shall have $u_{12} = u_{21} = 0$. If J vanishes, we must therefore have either u_{11} or u_{22} equal to zero. In either case, equations (2) prove that S' is developable.

If $\Theta_4 = 0$, the flecnode curve has only one branch, say C_y, so that $u_{12} = 0$. But from $\Theta_4 = J = 0$ then follows $u_{11} = u_{22} = 0$, so that again S' is developable.

In all cases then, if $J = 0$ the derivative of S with respect to x is developable. One may easily see that if $J \neq 0$, S' is not developable, for in that case the planes tangent to S' at P_ϱ and P_σ intersect the line L_{yz} joining P_y and P_z in distinct points, as is shown by equations (2).

We shall, therefore, obtain all developable surfaces of the congruence Γ, by finding the most general transformation $\xi = \xi(x)$, which reduces the seminvariant J to zero. But according to equation (51) of Chapter IV, the most general solution of the differential equation

$$(3) \qquad 4\{\xi, x\}^2 + 2I\{\xi, x\} + J = 0,$$

is the most general independent variable for which $J = 0$. To reduce J to zero we may, therefore, take for ξ the general solution of either of the two equations

$$(3a) \qquad \{\xi, x\} = \frac{-I + \sqrt{\Theta_4}}{4}, \qquad (3b) \quad \{\xi, x\} = -\frac{I - \sqrt{\Theta_4}}{4},$$

where

$$\Theta_4 = I^2 - 4J.$$

Although each of the equations (3a) and (3b) is of the third order, we obtain in this way only two families of ∞^1 developable surfaces, as we should. For, as has already been remarked, all values of ξ which give the same value to η belong to the same developable surface S'. As a matter of fact, equations (3a) and (3b) may be written

$$\eta' - \frac{1}{2}\eta^2 = \frac{-I + \sqrt{\Theta_4}}{4}, \quad \eta' - \frac{1}{2}\eta^2 = -\frac{I - \sqrt{\Theta_4}}{4},$$

and these are of the *Riccati* form, so that the anharmonic ratio of any four solutions is a constant.

We have therefore proved the following theorem:

To every ruled surface S there belongs a congruence Γ determined by its osculating hyperboloids. This congruence contains two families

of developable surfaces, which coincide if and only if $\Theta_4 = 0$, *i. e., if and only if the two branches of the flecnode curve of* S *coincide. To determine any developable surface of the congruence, it is necessary and sufficient to find a solution of the equation*

$$4\{\xi, x\}^2 + 2I\{\xi, x\} + J = 0,$$

and to take this solution $\xi = \xi(x)$ *as the independent variable of the defining system of differential equations. The derivative of* S *with respect to* ξ *will then be a developable surface, and all developable surfaces of the congruence may be obtained in this way. Moreover, any four developables of the same family intersect all of the asymptotic tangents of* S *in point rows of the same cross-ratio.*

Let us suppose that the variable x has been so chosen as to make $J = 0$. Then the line $L_{\varrho\sigma}$, joining P_ϱ and P_σ generates a developable surface of the congruence Γ, and C_ϱ and C_σ are two curves on this surface. Let us assume that $\Theta_4 \neq 0$, and that C_y and C_z are the two (distinct) branches of the flecnode curve on S. Then

$$u_{12} = u_{21} = 0, \quad u_{11} - u_{22} \neq 0, \quad J = u_{11}u_{22} - u_{12}u_{21} = 0,$$

so that either u_{11} or u_{22}, but not both, will vanish. Suppose that $u_{11} = 0$ Then, according to (2),

$$2\varrho' + p_{11}\varrho + p_{12}\sigma = 0,$$

i. e., if $p_{12} \neq 0$, P_σ is a point on the tangent to the curve C_ϱ described by P_ϱ. In other words, C_ϱ is the cuspidal edge of the developable surface. If p_{12}, together with u_{12}, were zero, C_y would be a straight line, and the curve C_ϱ would degenerate into a point of this line. The developable surface would be a cone

If $u_{11} \neq 0$, u_{22} must vanish, and then C_σ is the cuspidal edge of the developable surface. This ambiguity corresponds to the fact that every line of the congruence belongs to two of its developable surfaces.

But, if P_y and P_z describe the flecnode curves on S, P_ϱ and P_σ are points on the flecnode tangents. We have called the ruled surface of two sheets, generated by the flecnode tangents of S, its *flecnode surface*, so that we have proved the theorem: *the focal surfaces of the congruence* Γ *are the two sheets,* F' *and* F'', *of the flecnode surface of* S.

For, the focal surfaces of any congruence are the loci of the cuspidal edges of its developable surfaces. The theorem is true also if $\Theta_4 = 0$, only in that case F' and F'' coincide.[1])

1) For a clear treatment of the general properties of congruences, the reader may consult *Darboux* Théorie des surfaces, t. II, chapter 1. Compare also *Wilczynski*, Invariants of a system of linear partial differential equations, and the theory of congruences of rays American Jour. of Math. vol XXVI (1904) pp. 319—360.

We may also prove this theorem geometrically. Let g_0, g_1, g_2, g_3, etc., be consecutive generators of the ruled surface S. The hyperboloids H_1, H_2, etc., osculating S along g_1, g_2, etc., are determined respectively by g_0, g_1, g_2; g_1, g_2, g_3, etc. The flecnode tangents f_1', f_1'', along generator g_1, are the two straight lines intersecting g_0, g_1, g_2, g_3. The flecnode tangents f_2', f_2'' of S, along g_2, intersect g_1, g_2, g_3, g_4; f_3', f_3'' intersect g_2, g_3, g_4, g_5; etc. Therefore, g_3 intersects f_1', f_2', f_3', f_4' as well as $f_1'', f_2'', f_3'' f_4''$, i e, four consecutive generators of each sheet of the flecnode surface of S. This shows that each of the sheets F' and F'' of the flecnode surface of S, has S itself as one of the sheets of *its* flecnode surface. The congruence Γ is made up of the generators of the first kind on the hyperboloids H_1, H_2, etc H_1 and H_2 intersect along the four lines g_1, g_2, f_1', f_1''. H_2 and H_3 intersect along g_2, g_3, f_2', f_2'', etc. Therefore, a generator of the first kind on H_1 can meet a generator of the first kind on H_2 only along one of the lines f_1' or f_1''. Moreover, at every point of f_1' and f_1'' two such lines actually do meet. It is clear then, that the cuspidal edges of the developable surfaces of the congruence Γ must lie on one or the other of the two sheets of the flecnode surface of S. This completes the synthetic proof of our theorem.

But we have also seen that *each sheet of the flecnode surface of S, has S itself as one of the sheets of its flecnode surface.*

To prove this analytically as well, we recur to the system of differential equations for F', which was set up in Chapter VI, equations (15).

Denote by $\mathfrak{U}_{ik}$ the quantities formed from this system in the same way as are u_{ik} from (A). Then we shall find

$$(4) \qquad \mathfrak{U}_{12} = 0, \quad \mathfrak{U}_{21} = -4(q_{11}' - q_{22}') + 8(q_{11} - q_{22})\frac{q_{12}}{p_{12}},$$
$$\mathfrak{U}_{11} - \mathfrak{U}_{22} = 4(q_{11} - q_{22}),$$

which equations are also valid for $\Theta_4 = 0$, in which case $q_{22} = q_{11}$.

Therefore, the curve C_y is flecnode curve on F' as well as on S. *If the two branches of the flecnode curve are distinct on S, they are also distinct on F' and F'',* for $q_{11} - q_{22}$ does not vanish under our assumptions unless $\Theta_4 = 0$ To complete the proof that S is a sheet of the flecnode surface of F', we have still to show that the flecnode tangents to F'' constructed, of course, along C_y are the generators of S. But the flecnode tangent to the surface F' at a point P_y of the flecnode curve, is the line joining it to the point, whose coordinates are obtained from system (15) Chapter VI in the same way as the coordinates of P_ϱ are obtained from system (A). But the coordinates of this point are

$$2y' - \frac{2q_{12}}{p_{12}} y - \varrho = -2\frac{q_{12}}{p_{12}} y - p_{12} z,$$

i. e., this point is on the generator of the surface S which passes through P_y. This completes the proof of our theorem if $p_{12} \neq 0$. But if p_{12} vanishes, together with u_{12}, F' degenerates into a straight line, and any ruled surface made up of lines intersecting it may be called its flecnode surface. The theorem may, therefore, be regarded as valid in all cases.

The curve C_y is one branch of the flecnode curve of F'. The other may be found by putting

(5) $$Y = \mathfrak{U}_{21} y - (\mathfrak{U}_{11} - \mathfrak{U}_{22})\varrho,$$

this being the second factor of the covariant C of the surface F'.

The flecnode tangents to F' along this curve generate the second sheet of the flecnode surface of F'; but this does not in general belong to the congruence Γ, —never in fact, as we shall see, unless F' degenerates into a straight line.

The flecnode surface F' may be of the second order. This is so if, and only if,

$$\mathfrak{U}_{12} = \mathfrak{U}_{21} = \mathfrak{U}_{11} - \mathfrak{U}_{22} = 0.$$

But this requires $q_{11} - q_{22}$ to vanish, which condition, together with those already fulfilled, gives

$$u_{12} = u_{21} = u_{11} - u_{22} = 0,$$

i. e., *only if the ruled surface S is of the second order, can a sheet of its flecnode surface be of the second order.* Its flecnode surface in that case is the surface itself, generated however by the generators of the second set.

If we put

(6) $$\begin{aligned} u_{21} v_{12} - u_{22} v_{11} &= 2J\lambda_{11}, & u_{21} v_{22} - u_{22} v_{21} &= 2J\lambda_{21}, \\ -u_{11} v_{12} + u_{12} v_{11} &= 2J\lambda_{12}, & -u_{11} v_{22} + u_{12} v_{21} &= 2J\lambda_{22}, \end{aligned}$$

we have seen, in Chapter VI, that the equations of the surface S' become

(7) $$\begin{aligned} \varrho'' + P_{11}\varrho' + P_{12}\sigma' + Q_{11}\varrho + Q_{12}\sigma &= 0, \\ \sigma'' + P_{21}\varrho' + P_{22}\sigma' + Q_{21}\varrho + Q_{22}\sigma &= 0, \end{aligned}$$

where

$$P_{ik} = p_{ik} + \lambda_{ik},$$

(8) $$\begin{aligned} Q_{11} &= q_{11} + \frac{1}{2}(\lambda_{11}p_{11} + \lambda_{12}p_{21}), & Q_{21} &= q_{21} + \frac{1}{2}(\lambda_{21}p_{11} + \lambda_{22}p_{21}), \\ Q_{12} &= q_{12} + \frac{1}{2}(\lambda_{11}p_{12} + \lambda_{12}p_{22}), & Q_{22} &= q_{22} + \frac{1}{2}(\lambda_{21}p_{12} + \lambda_{22}p_{22}). \end{aligned}$$

Let us denote by U_{ik} the quantities which are formed from system (7) in the same way as are the quantities u_{ik} from (A) Then we shall find

$$
\begin{aligned}
J U_{12} &= J u_{12} + \frac{1}{2}(u_{12} w_{11} - u_{11} w_{12}) - 3J' \lambda_{12}, \\
(9) \qquad J U_{21} &= J u_{21} + \frac{1}{2}(u_{21} w_{22} - u_{22} w_{21}) - 3J' \lambda_{21}, \\
J(U_{11} - U_{22}) &= J(u_{11} - u_{22}) + \frac{1}{2}(u_{21} w_{12} - u_{12} w_{21} + u_{11} w_{22} - u_{22} w_{11}) \\
&\quad - 3J'(\lambda_{11} - \lambda_{22}).
\end{aligned}
$$

If we assume $u_{12} = u_{21} = 0$, $\Theta_4 \neq 0$, the curves C_y and C_z are the flecnode curves on S, and the curves C_ϱ and C_σ, on the derived surface S', are the intersections of this surface with the flecnode surface of S. It may happen that C_ϱ is one branch of the flecnode curve of S'. This is so if, and only if, $U_{12} = 0$. In other words, the derivative of S has a branch of its flecnode curve on the flecnode surface of S, if one of the two conditions

$$u_{12} = U_{12} = 0 \quad \text{or} \quad u_{21} = U_{21} = 0$$

is satisfied.

There exist two families of ∞^2 non-developable ruled surfaces of the congruence Γ each of which has one branch of its flecnode curve on the flecnode surface of S.

This we shall now proceed to prove. Assuming $J \neq 0$, $U_{12} = 0$ will be a consequence of $u_{12} = 0$, if

$$(10) \qquad w_{12} - 3 v_{12} \frac{J'}{J} = 0,$$

as is shown by (9). The equation $u_{12} = 0$ is left invariant by an arbitrary transformation of the independent variable. If then C_y be taken as one branch of the flecnode curve on S, the curve C_ϱ will be a branch of the flecnode curve on S', provided the independent variable be so chosen as to satisfy the equation

$$w_{12} - 3 v_{12} \frac{J'}{J} = 0,$$

or, making use of (49), (53) and (54) of Chapter IV, if

$$(11) \qquad w_{12} + 2 v_{12} \eta - 3 v_{12} \frac{J' + 2\{\xi, x\} I' + 2\{\xi, x\}' I + 8\{\xi, x\}\{\xi, x\}'}{J + 2\{\xi, x\} I + 4\{\xi, x\}^2} = 0,$$

which equation is of the second order with respect to η. This proves our assertion, that there exists a family of ∞^2 ruled surfaces in the congruence Γ, each such that one of the branches of its flecnode curve lies on F'. There is another such family connected with the other sheet F'' of the flecnode surface. The surfaces of the second family are determined by the equation

$$(12) \qquad w_{21} + 2 v_{21} \eta - 3 v_{21} \frac{J' + 2\{\xi, x\} I' + 2\{\xi, x\}' I + 8\{\xi, x\}\{\xi, x\}'}{J + 2\{\xi, x\} I + 4\{\xi, x\}^2} = 0,$$

which is obtained from (11) by permuting the indices 1 and 2.

It may happen that the derivative of S has both of the branches of its flecnode curve on the flecnode surface of S, one on each sheet. If this is so (assuming $\Theta_4 \neq 0$), we have simultaneously

$$u_{12} = u_{21} = U_{12} = U_{21} = 0,$$

whence

$$w_{12} - 3v_{12}\frac{J'}{J} = 0, \quad w_{21} - 3v_{21}\frac{J'}{J} = 0.$$

But, since J is not zero, this gives

$$w_{12}v_{21} - w_{21}v_{12} = 0,$$

which together with $u_{12} = u_{21} = 0$, makes

$$\varDelta = \begin{vmatrix} u_{11} & -u_{22} & u_{12} & u_{21} \\ v_{11} & -v_{22} & v_{12} & v_{21} \\ w_{11} & w_{22} & w_{12} & w_{21} \end{vmatrix} = 0$$

But this is the condition under which S belongs to a linear complex (Chapter VIII). The converse is also true, i. e., if $\varDelta = 0$ and $\Theta_4 \neq 0$, a double infinity of surfaces S' exists, each of which has the property in question. For, equations (11) and (12), will then be identical, the two (distinct) branches of the flecnode curve on S being taken as fundamental curves.

If however Θ_4 vanishes, we may still assume $u_{12} = 0$, whence follows in this case $u_{11} - u_{22} = 0$. The flecnode surface F of S has only one sheet, and in order that both of the branches of the flecnode curve of S' may be on F, they must coincide. We must then, in this case, choose the independent variable so as to satisfy the simultaneous conditions

$$u_{12} = U_{12} = u_{11} - u_{22} = U_{11} - U_{22} = 0.$$

The first three give

$$w_{12} = -2p_{12}^2 u_{21} = 0,$$

so that either u_{21} or p_{12} must vanish, i. e. S is either a quadric or has at least a straight line directrix. In either case, the other condition will also be satisfied, without specializing the independent variable in any way. In other words, not only a double infinity, but all ruled surfaces of Γ will have the property in question. The same is true for $\Theta_4 \neq 0$, if $v_{12} = v_{21} = 0$.

We have the following theorem:

If there exists a ruled surface of the flecnode congruence Γ, different from S, which has both of the branches of its flecnode curve on the flecnode surface of S, one on each sheet, then the surface S belongs to a linear complex.

Conversely, if S belongs to a linear complex, but not to a linear congruence, there will be ∞^2 ruled surfaces in its flecnode congruence

which have the required property. If S belongs to a linear congruence, this is true not merely of ∞^2 but of all ruled surfaces of the flecnode congruence which, in this case, coincides with the linear congruence to which S belongs.

If Δ does not vanish identically, it will, in general, vanish for particular values of x. If the flecnodes on a generator of S, corresponding to a particular value of $x = a$ for which $\Delta = 0$, are distinct, one of two things must take place Either the osculating hyperboloid hyperosculates the surface along that generator, or else the two flecnodes corresponding to $x = a$ on the derivative are on the flecnode surface of S, one on each sheet. For if we take $u_{12} = u_{21} = 0$, the condition that Δ vanishes gives either

$$u_{12} = u_{21} = u_{11} - u_{22} = 0, \quad \text{or} \quad u_{12} = u_{21} = v_{12} w_{21} - v_{21} w_{12} = 0,$$

for $x = a$

In the first case, the osculating hyperboloid hyperosculates the surface. In the second case, any solution of (11) will, for $x = a$, also satisfy (12), i. e, for $x = a$ the simultaneous conditions

$$u_{12} = U_{12} = u_{21} = U_{21} = 0$$

will be satisfied, i e., P_ϱ and P_σ, two points on the flecnode surface of S, will be on the flecnode curve of S'. One sees at once how this is to extended to the case when $\Theta_4 = 0$.

Equation (11) can be integrated once. Since we have $u_{12} = u_{21} = 0$, we find [Chapter IV, equations (32) and (39)];

$$v_{12} = - p_{12}(u_{11} - u_{22}), \quad v_{11} - v_{22} = 2(u'_{11} - u'_{22}),$$
$$w_{12} = 2v'_{12} + (p_{11} - p_{22}) v_{12} - p_{12}(v_{11} - v_{22}).$$

Divide both members of (11) by v_{12}. We find

$$2\frac{v'_{12}}{v_{12}} + p_{11} - p_{22} + 2\frac{u'_{11} - u'_{22}}{u_{11} - u_{22}} + 2\frac{\xi''}{\xi'}$$
$$- 3\frac{d \log}{dx}[J + 2\{\xi, x\} I + 4\{\xi, x\}^2] = 0.$$

Moreover, since $u_{12} = u_{21} = 0$, we have

$$I = u_{11} + u_{22}, \quad J = u_{11} u_{22}.$$

Therefore, we obtain by integration

$$\text{(13)} \qquad \frac{p_{12}^2 (u_{11} - u_{22})^4 (\xi')^2}{(u_{11} + 2\{\xi, x\})^3 (u_{22} + 2\{\xi, x\})^3} e^{\int (p_{11} - p_{22}) dx} = c,$$

where c is a constant. Of course (12) may be treated in the same manner.

We have seen that there exist ∞^2 ruled surfaces of the congruence Γ, the flecnode curve of each of which has one of its branches

upon one of the sheets of the flecnode surface of S. The question arises, whether among these there exists one (there cannot in general be more than one) whose flecnode surface has one of its sheets in common with that of S.

Let us suppose that F' is at the same time a sheet of the flecnode surface of S and of S'. Then C_y and C_ϱ are flecnode curves on S and S' respectively, so that we shall have $u_{12} = U_{12} = 0$. But more than that, the flecnode tangent to S' at any point of C_ϱ must be a generator of F', i. e., must pass through P_y and P_ϱ. In other words, the conditions

$$u_{12} = U_{12} = 0, \quad 2\varrho' + P_{11}\varrho + P_{12}\sigma = \lambda y + \mu\varrho$$

must be simultaneously fulfilled, where λ and ϱ are some (as yet unknown) functions of x. But we have

$$P_{ik} = p_{ik} + \lambda_{ik},$$

$$2\varrho' + p_{11}\varrho + p_{12}\sigma = u_{11}y + u_{12}z,$$

so that our second condition becomes

$$u_{11}y + \lambda_{11}\varrho + \lambda_{12}\sigma = \lambda y + \mu\varrho,$$

or

$$(\lambda_{11} - \mu)\varrho + \lambda_{12}\sigma = (\lambda - u_{11})y$$

But, except for singular values of x, P_y, P_ϱ and P_σ are not collinear. Therefore we must have

$$\mu = \lambda_{11}, \quad \lambda = u_{11}, \quad \lambda_{12} = 0,$$

which last equation, together with $u_{12} = 0$, gives $v_{12} = 0$ if $J \neq 0$. But $u_{12} = v_{12} = 0$ is satisfied by either $p_{12} = 0$, in which case F' degenerates into a straight line, or by

$$u_{12} = u_{11} - u_{22} = 0.$$

But since U_{12} must also vanish, we must have also $w_{12} = 0$, which, if $p_{12} \neq 0$, gives the additional condition $u_{21} = 0$, i. e., S must be a surface of the second order

We have proved the following theorem. *If the surface S is not of the second order, and if its flecnode surface has a sheet F' which does not degenerate into a straight line, no other ruled surface of the congruence Γ has F' also as a sheet of its flecnode surface Or, in other words, the second sheet of the flecnode surface of F' does not belong to the congruence Γ.*

If F' does degenerate into a straight line, every ruled surface of Γ clearly has this straight line as a degenerate sheet of its flecnode surface.

We have seen that F' may degenerate into a straight line. It is, in general, a ruled surface. Can it be a developable surface? If it were developable, according to equations (15) of Chapter VI, p_{12} would have to vanish. But the simultaneous conditions $p_{12} = u_{12} = 0$, would make C_y a straight line. Therefore: *if a sheet of a flecnode surface is developable, it degenerates into a straight line.*

This is a generalization of the result which we found in Chapter VI; that the flecnode tangent is tangent to the flecnode curve only if the latter is a straight line. For, in that case, the flecnode curve would be also an asymptotic curve, and the flecnode tangents along it would form a developable.

§ 2. Correspondence between the curves on a ruled surface and on its derivative.

Assuming again $u_{12} = u_{21} = 0$, we find

$$P_{12} = p_{12} + \frac{1}{2u_{11}u_{22}}(-u_{11}v_{12}) - \frac{p_{12}}{2u_{22}}(u_{11} + u_{22}),$$

$$P_{21} = p_{21} + \frac{1}{2u_{11}u_{22}}(-u_{22}v_{21}) - \frac{p_{21}}{2u_{11}}(u_{11} + u_{22}).$$

But we can always choose the independent variable so as to make $u_{11} + u_{22} = I$ vanish. According to (51) of Chapter VI, it is only necessary, for this purpose, to take as the new independent variable any solution of the equation

$$(14) \qquad 4\{\xi, x\} + I = 0,$$

which gives the condition

$$(15) \qquad 2(2\eta' - \eta^2) + I = 0$$

for η. This is of the first order and of the *Riccati* form, so that the anharmonic ratio of any four solutions is constant. The conditions $P_{12} = P_{21} = 0$ prove that C_ϱ and C_σ are asymptotic lines on S'. Moreover, if p_{12} and p_{21} are not both zero in the above equations, i. e., if S is not contained in a linear congruence, it is not only sufficient, but it is necessary to make $u_{11} + u_{22}$ vanish so as to have $P_{12} = P_{21} = 0$. We have the following theorem:

If S is a ruled surface with two distinct branches to its flecnode curve and not belonging to a linear congruence, there exists just a single infinity of ruled surfaces in the congruence Γ, whose intersections with the two sheets of the flecnode surface of S are asymptotic lines upon them. They are the derivatives of S, when the independent variable is so chosen as to make the seminvariant I vanish. Moreover, the point-rows, in which any four of these surfaces intersect the asymptotic tangents of S, all have the same anharmonic ratio.

In Chapter IV, § 6, we considered a canonical form, to which system (A) can always be reduced, namely that, for which

$$p_{ik} = 0, \quad q_{11} + q_{22} = 0.$$

We can now say that, if a system (A) *is written in its canonical form, its integral curves are asymptotic lines on its integrating ruled surface, and its derivative with respect to x is cut by the two sheets of the flecnode surface of S along asymptotic lines.*

If $\Theta_4 = 0$, since

$$\Theta_4 = I^2 - 4J,$$

J also will vanish for the canonical form, so that *in this case the surfaces just mentioned will coincide with the (single) family of developable surfaces of the congruence.*

If $p_{12} = p_{21} = 0$ together with $u_{12} = u_{21} = 0$, the flecnode curve of S consists of two straight lines, and every ruled surface of the congruence has the property in question The reduction to the canonical form must, therefore, have a different significance in this case. To find it, let us assume that the curves C_y and C_z are asymptotic lines on S, but not at the same time flecnode curves, i. e., let them be any two asymptotic lines different from the straight line directrices. Then p_{12} and p_{21} will vanish, while u_{12} and u_{21} do not. We may, moreover, also assume $p_{11} = p_{22} = 0$, so that system (A) has been reduced to the semi-canonical form.

The conditions, which are necessary and sufficient to make S belong to a linear congruence, are that all of the minors of the second order in Δ must vanish. Three of these minors are

$$(u_{11} - u_{22})\, v_{12} - u_{12}\,(v_{11} - v_{22}), \quad (u_{11} - u_{22})\, v_{21} - u_{21}\,(v_{11} - v_{22}),$$

$$u_{12}\, v_{21} - u_{21}\, v_{12}.$$

Since we have assumed $p_{ik} = 0$,

$$u_{ik} = -4q_{ik}, \quad v_{ik} = -8q'_{ik}, \quad w_{ik} = -16q''_{ik}.$$

Inserting these values in the above minors of Δ, and equating them to zero, we find that the ratios $q_{12} : q_{21} : q_{11} - q_{22}$ must be constants. If they are, the other minors will also vanish.

We may, therefore, put

$$q_{12} = aq, \quad q_{21} = bq, \quad q_{11} - q_{22} = cq, \tag{16}$$

a, b, and c being constants, so that our system (A) has the form

$$\begin{aligned} y'' + q_{11}y + aqz &= 0, \\ z'' + bqy + q_{22}z &= 0, \end{aligned} \quad q_{11} - q_{22} = cq. \tag{17}$$

If we compute the coefficients P_{12} and P_{21} of the derived system from (8), we shall find

(18) $$P_{12} = a\,\frac{-q_{11}q' + qq'_{11}}{q_{11}q_{22} - q_{12}q_{21}}, \qquad P_{21} = b\,\frac{-q_{22}q' + qq'_{22}}{q_{11}q_{22} - q_{12}q_{21}}.$$

P_{12} will be zero, if and only if the ratio $q_{11}:q$ is a constant, i. e. if

$$q_{11} = \lambda t, \quad q = \mu t,$$

where λ and μ are constants. We shall then have

$$q_{12} = a\mu t, \quad q_{21} = b\mu t, \quad q_{11} \quad \lambda t, \quad q_{22} = (\lambda - c\mu)t,$$

so that P_{21} also is zero. If, therefore, the independent variable be chosen in such a way that there is a linear relation between q_{11} and q_{22} of the form

(19) $$\alpha q_{11} + \beta q_{22} = 0,$$

where α and β are constants, whose ratio only is of importance, the curves C_ϱ and C_σ on S', which correspond to the curves C_y and C_z on S, will be asymptotic curves.[1]) But (19) is the special form to which the equation

(20) $$\alpha u_{11} + \beta u_{22} = 0$$

reduces when (A) is reduced to its semi-canonical form, and $\frac{\alpha}{\beta}$ is an essential constant. If $\alpha u_{11} + \beta u_{22}$ is not zero, the transformation $\xi = \xi(x)$ makes it zero, if $\xi(x)$ satisfies the equation of the *Riccati* form,

$$\alpha u_{11} + \beta u_{22} + 2(\alpha + \beta)\left(\eta' - \frac{1}{2}\eta^2\right) = 0, \quad \eta = \frac{\xi''}{\xi'}.$$

This equation is of the first order, but contains in its coefficients an arbitrary constant $\frac{\alpha}{\beta}$. We find, therefore, ∞^2 solutions for η, i. e. there are ∞^2 ruled surfaces of the congruence which have the property that two of their asymptotic curves correspond to two of the asymptotic curves of S.

But we see that, in this case, all of the asymptotic curves of S' correspond to those of S. Therefore: *if a ruled surface S is contained in a linear congruence, there exists a double infinity of ruled surfaces in this congruence whose asymptotic lines correspond to those of S. They are obtained by taking the derivative of S with respect to an independent variable, which is chosen so as to make*

$$\alpha u_{11} + \beta u_{22} = 0,$$

where $\alpha:\beta$ is an arbitrary constant.

---

1) If $c \neq 0$, the ratio $\alpha:\beta$ cannot be equal to -1, since the equation $q_{11} - q_{22} = 0$ remains invariant for all transformations $\xi = \xi(x)$. We may always assume $c \neq 0$, so that the curves C_y, C_z and the directrices do not divide the generators harmonically.

Let the independent variable be so chosen. The planes tangent to S' at P_ϱ and P_σ intersect L_{yz} in the points P_u and P_v respectively, where

$$u = q_{11} y + aqz, \quad v = bqy + q_{22} z,$$

[cf. Chapter VI, equ. (2)]. Under the condition (20) the curves C_u and C_v described by P_u and P_v are again asymptotic curves. According to *Serret's* theorem, the anharmonic ratio of these four curves is constant. It is found to be

$$-\frac{ab}{c^2}\frac{(\alpha+\beta)^2}{\alpha\beta},$$

a function of the ratio $\alpha:\beta$. For the canonical form this double ratio becomes

$$-\frac{4\,ab}{c^2}.$$

The ∞^2 surfaces of the congruence, whose asymptotic curves correspond to those of S, may therefore be arranged in a single infinity of one-parameter families. Each family is characterized by the value of the ratio $\alpha:\beta$, i. e. by the double-ratio of the four asymptotic curves C_y, C_z, C_u, C_v. Any four surfaces of the same family intersect all of the asymptotic tangents of S in point-rows of the same cross-ratio.

The significance of the reduction to the canonical form has now been made clear, also in this case.

Let us suppose that the asymptotic curves C_ϱ and C_σ of S' correspond to C_y and C_z, and that the differential equations for S' assume the semi-canonical form simultaneously with those for S. In other words, let the simultaneous conditions

$$p_{ik} = P_{ik} = 0$$

be verified. Then we shall have

$$\lambda_{ik} = 0,$$

whence, since $\varDelta \neq 0$, the surface S' not being developable,

$$v_{ik} = 0,$$

so that the quantities u_{ik} and, therefore, q_{ik} are constants. All the minors of the second order in $\varDelta$ will vanish. We have, therefore, a special case of a ruled surface S contained in a linear congruence, determined by a system of equations

$$y'' + q_{11} y + q_{12} z = 0, \quad z'' + q_{21} y + q_{22} z = 0, \tag{21}$$

where the quantities q_{ik} are constants. Conversely, if (A) has the form (21), the equations of the derivative surface S' are actually in the semi-canonical form. But more than that, the differential equations of S' are *identical* with those of S, so that *the surfaces S and S' are projective transformations of each other.*

Denote by S'' the derivative of S' with respect to x, the *second derivative* of S. Then the generator g'' of S'' joins the points

$$(22)\qquad \begin{aligned} 2\varrho' + P_{11}\varrho + P_{12}\sigma &= u_{11}y + u_{12}z + \lambda_{11}\varrho + \lambda_{12}\sigma, \\ 2\sigma' + P_{21}\varrho + P_{22}\sigma &= u_{21}y + u_{22}z + \lambda_{21}\varrho + \lambda_{22}\sigma, \end{aligned}$$

obtained from the equations of S' in the same way as are ϱ and σ from the equations of S. But clearly, g'' coincides with g if and only if $\lambda_{ik} = 0$. Since we assume $J \neq 0$, these conditions are equivalent to $v_{ik} = 0$. These equations form a seminvariant system; i. e. if they are satisfied for a pair of curves C_y and C_z upon S, they are fulfilled for any other pair of curves on S, the independent variable not being transformed. The conditions $p_{ik} = 0$ may be satisfied without transforming the independent variable. We shall then find also $P_{ik} = 0$. *Every ruled surface S, which has a second derivative S'' coinciding with S itself, may therefore be defined by a system of form (21) with constant coefficients. If the independent variable is so chosen that its second derivative S'' coincides with S, the first derivative S' is a projective transformation of the original surface, and its asymptotic lines correspond to those of S.*

We proceed to find the explicit equations of these ruled surfaces

Let us assume that the two straight line directrices of S are distinct. Since they will be obtained by factoring the covariant C, whose coefficients in this case are constants, we may write (21) in the form

$$(23)\qquad y'' + q_{11}y = 0, \quad z'' + q_{22}z = 0,$$

where q_{11} and q_{22} are non-vanishing constants, and where

$$q_{11} - q_{22} \neq 0,$$

since we have assumed $J \neq 0$ and $\Theta_4 \neq 0$. The integral curves of (23) are the straight line directrices of S. Let the edge $x_3 = x_4 = 0$ of the fundamental tetrahedron of reference coincide with C_y, while the edge $x_1 = x_2 = 0$ coincides with C_z. We may then put

$$\begin{aligned} y_1 &= e^{r_1 x}, & y_2 &= e^{r_2 x}, & y_3 &= 0, & y_4 &= 0, \\ z_1 &= 0, & z_2 &= 0, & z_3 &= e^{r_3 x}, & z_4 &= e^{r_4 x}, \end{aligned}$$

where

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \pm\sqrt{-q_{11}}, \quad \left.\begin{matrix} r_3 \\ r_4 \end{matrix}\right\} = \pm\sqrt{-q_{22}}.$$

If we put

$$x_k = \alpha y_k + \beta z_k,$$

where α and β are arbitrary functions of x we shall obtain $(x_1, \dots x_4)$ as the coordinates of an arbitrary point of the surface. We find

$$x_1 = \alpha e^{r_1 x}, \quad x_2 = \alpha e^{r_2 x}, \quad x_3 = \beta e^{r_3 x}, \quad x_4 = \beta e^{r_4 x}.$$

The relation between $x_1, \dots x_4$, obtained by eliminating α, β, x from these equations, will be the equation of the ruled surface S. It is

$$x_1^{2\sqrt{-q_{22}}} x_4^{2\sqrt{-q_{11}}} - x_2^{2\sqrt{-q_{22}}} x_3^{2\sqrt{-q_{11}}} = 0,$$

or more briefly

$$x_1^{\lambda} x_4^{\mu} - x_2^{\lambda} x_3^{\mu} = 0, \tag{24}$$

where λ and μ are constants.

If the directrices of the congruence coincide, we may write (21) in the form

$$y'' + q_{11} y = 0, \quad z'' + q_{21} y + q_{11} z = 0, \quad q_{11} \neq 0, \tag{25}$$

where C_y is the straight line directrix, which we shall again identify with the edge $x_3 = x_4 = 0$ of the tetrahedron of reference. We may therefore put

$$y_1 = e^{+rx}, \quad y_2 = e^{-rx}, \quad y_3 = 0, \quad y_4 = 0,$$

where

$$r = \sqrt{-q_{11}}.$$

We shall then find

$$z_1 = \frac{q_{21}}{2r}\left(\frac{1}{2r} - x\right) e^{rx}, \quad z_2 = \frac{q_{21}}{2r}\left(\frac{1}{2r} + x\right) e^{-rx}, \quad z_3 = e^{rx}, \quad z_4 = e^{-rx},$$

so that

$$x_1 = \left[\alpha + \beta \frac{q_{21}}{2r}\left(\frac{1}{2r} - x\right)\right] e^{rx},$$

$$x_2 = \left[\alpha + \beta \frac{q_{21}}{2r}\left(\frac{1}{2r} + x\right)\right] e^{-rx},$$

$$x_3 = \beta e^{rx}, \quad x_4 = \beta e^{-rx}$$

will be the coordinates of an arbitrary point of the surface. By elimination we find

$$2 q_{11} (x_1 x_4 - x_2 x_3) = q_{21} x_3 x_4 \log \frac{x_3}{x_4} \tag{26}$$

as the equation of the surface.

From (21) we find that y and z separately satisfy the differential equations

$$y^{(4)} + (q_{11} + q_{22}) y'' + (q_{11} q_{22} - q_{12} q_{21}) y = 0,$$

$$z^{(4)} + (q_{11} + q_{22}) z'' + (q_{11} q_{22} - q_{12} q_{21}) z = 0.$$

Since any asymptotic line may be found by putting

$$u = ay + bz,$$

where a and b are constants, we see that all of the asymptotic curves of the surface satisfy the same linear differential equation of the fourth order with constant coefficients. We shall, in the theory of space curves, speak of such curves as anharmonic curves, and may therefore

express our result as follows. *Every ruled surface, which has a second derivative coinciding with itself, has the property that all of its asymptotic lines are anharmonic curves with the same invariants.*

Examples.

Ex. 1. Prove that for every ruled surface S there exist ∞^2 derivative surfaces S', such that one asymptotic curve on S' corresponds to one on S. Find the condition that there may be two asymptotic curves on S and S' which correspond to each other.

Ex. 2. Find the conditions that a branch of a flecnode curve of S' may correspond to an asymptotic curve of S; that both branches of the flecnode curve on S' may so correspond to asymptotic curves of S.

Ex. 3. According to the general theory of congruences, the two families of developable surfaces intersect F' and F'' along conjugate curves. Prove this directly.

Ex. 4.* The cuspidal edges of one of the families of developables form a family of ∞^1 curves on F'. Consider the family of ∞^1 curves conjugate to the first. Its tangents will form a congruence, one of the sheets of whose focal surface is F'. Find the other sheet. Similarly for F''.

Remark. This problem is a geometrical formulation for the *Laplace transformation* of a partial differential equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0,$$

for which cf. *Darboux, Théorie des Surfaces*, vol. II, Chapter I. The repetition of this transformation will give a series of covariants, in general infinite in number. An extensive theory is thus suggested. In particular the question arises whether these surfaces thus obtained from F' and F'' are, in general, ruled surfaces, and if not, under what conditions they will be.

Ex. 5.* Another question, closely related to the preceding problem, concerns the general theory of congruences. We have found a congruence Γ, whose focal surface is a ruled surface. We suggest the general problem: to investigate the properties of congruences whose focal surfaces are ruled. For the case of a W-congruence, which is at the same time a congruence of normals, this problem may be solved without difficulty (cf. *Bianchi-Lukat.* Vorlesungen über Differentialgeometrie, Chapter IX).

CHAPTER X.

THE FLECNODE CONGRUENCE (CONTINUED).

§ 1. The derivative cubic curve.

If α_1 and α_2 are arbitrary, $\alpha_1 y_k + \alpha_2 z_k$ will represent the coordinates of an arbitrary point on the generator g of the ruled surface, where (y_k, z_k) for $k = 1, 2, 3, 4$ are four simultaneous systems of solutions of our system of differential equations, whose determinant does not vanish. We shall usually write $\alpha_1 y + \alpha_2 z$, suppressing the index k, as has been done occasionally in previous chapters. Of course, this is essentially a form of vector analysis, which enables us to make one equation do the work of four. The point $\alpha_1 \varrho + \alpha_2 \sigma$ of the corresponding generator g' of S', will then be such, that the line joining it to $\alpha_1 y + \alpha_2 z$ is a generator of the second kind on the hyperboloid H osculating S along g. Therefore, if β_1 and β_2 are arbitrary,

$$\beta_1 (\alpha_1 y + \alpha_2 z) + \beta_2 (\alpha_1 \varrho + \alpha_2 \sigma)$$

will be an arbitrary point of H.

If we choose the tetrahedron $P_y P_z P_\varrho P_\sigma$ as tetrahedron of reference, we may choose the homogeneous coordinates in such a way, that an expression of the form $\lambda y + \mu z + \nu \varrho + \varkappa \sigma$ will represent the point $x_1 = \lambda$, $x_2 = \mu$, $x_3 = \nu$, $x_4 = \varkappa$. We have then

$$x_1 = \beta_1 \alpha_1, \quad x_2 = \beta_1 \alpha_2, \quad x_3 = \beta_2 \alpha_1, \quad x_4 = \beta_2 \alpha_2$$

as the coordinates of an arbitrary point on H, and therefore

$$x_1 x_4 - x_2 x_3 = 0 \tag{1}$$

as the equation of H in this system of coordinates.

Let us consider now the hyperboloid H', which osculates S' along g'. The coordinates of P_ϱ and P_σ were obtained from the system of differential equations defining S by forming

$$\varrho = 2y' + p_{11} y + p_{12} z, \quad \sigma = 2z' + p_{21} y + p_{22} z.$$

We shall obtain the coordinates of two points on a generator g'' of the derivative of S' with respect to x, by applying the same process to the equations (8) of Chapter VI, which define S'. The ruled surface S'', thus obtained, shall be called the second derivative of S with respect to x. Its generator g'' is then a generator of the hyperboloid H' which osculates S' along g'. The following quantities

$$\begin{aligned} 2\varrho' + P_{11} \varrho + P_{12} \sigma &= u_{11} y + u_{12} z + \lambda_{11} \varrho + \lambda_{12} \sigma, \\ 2\sigma' + P_{21} \varrho + P_{22} \sigma &= u_{21} y + u_{22} z + \lambda_{21} \varrho + \lambda_{22} \sigma \end{aligned} \tag{2}$$

are the coordinates of two points on g''.

These equations show that g'' intersects g, if and only if $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0$, i. e if $K = v_{11}v_{22} - v_{12}v_{21} = 0$, provided we assume that S' is not developable, so that $J \neq 0$. By changing the independent variable one can always change K into $\overline{K}$ such that $\overline{K} = 0$. The differential equation, which η must satisfy so as to make $\overline{K} = 0$, is of the second order. *Therefore, there exist ∞^2 non-developable ruled surfaces in the congruence Γ, each of which, when considered as the first derivative of S, gives rise to a second derivative whose generators intersect the corresponding generators of S.*

Let us consider any point on g', whose coordinates are $\varepsilon_1\varrho + \varepsilon_2\sigma$. The corresponding point on g'' will be given by

$$(\varepsilon_1 u_{11} + \varepsilon_2 u_{21})y + (\varepsilon_1 u_{12} + \varepsilon_2 u_{22})z + (\varepsilon_1\lambda_{11} + \varepsilon_2\lambda_{21})\varrho + (\varepsilon_1\lambda_{12} + \varepsilon_2\lambda_{22})\sigma$$

Therefore, the expression

$$\begin{aligned}\delta_1(\varepsilon_1 u_{11} + \varepsilon_2 u_{21})y &+ \delta_1(\varepsilon_1 u_{12} + \varepsilon_2 u_{22})z \\ &+ [\delta_1(\varepsilon_1\lambda_{11} + \varepsilon_2\lambda_{21}) + \delta_2\varepsilon_1]\varrho + [\delta_1(\varepsilon_1\lambda_{12} + \varepsilon_2\lambda_{22}) + \delta_2\varepsilon_2]\sigma\end{aligned}$$

will, for arbitrary values of $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$, represent an arbitrary point of H'. If we introduce again our special system of coordinates, we have

$$\begin{aligned} x_1 &= u_{11}\delta_1\varepsilon_1 + u_{21}\delta_1\varepsilon_2, & x_3 &= \lambda_{11}\delta_1\varepsilon_1 + \lambda_{21}\delta_1\varepsilon_2 + \delta_2\varepsilon_1, \\ x_2 &= u_{12}\delta_1\varepsilon_1 + u_{22}\delta_1\varepsilon_2, & x_4 &= \lambda_{21}\delta_1\varepsilon_1 + \lambda_{22}\delta_1\varepsilon_2 + \delta_2\varepsilon_2, \end{aligned}$$

as the coordinates of an arbitrary point of H'. If we eliminate $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2$, we find *the equation of H'*:

$$\begin{aligned}(3)\qquad &(u_{22}x_1 - u_{21}x_2)[Jx_4 - (\lambda_{12}u_{22} - \lambda_{22}u_{12})x_1 + (\lambda_{12}u_{21} - \lambda_{22}u_{11})x_2] \\ &+ (u_{12}x_1 - u_{11}x_2)[Jx_3 - (\lambda_{11}u_{22} - \lambda_{21}u_{12})x_1 + (\lambda_{11}u_{21} - \lambda_{21}u_{11})x_2] = 0.\end{aligned}$$

We shall mostly assume that P_y and P_z are the flecnodes of g, supposed distinct, so that $u_{12} = u_{21} = 0$, and (3) simplifies into

$$\begin{aligned}(3a)\qquad -\lambda_{12}u_{22}^2x_1^2 + \lambda_{21}u_{11}^2x_2^2 &+ u_{11}u_{22}(\lambda_{11} - \lambda_{22})x_1x_2 \\ &+ u_{11}u_{22}(u_{22}x_1x_4 - u_{11}x_2x_3) = 0.\end{aligned}$$

It is easy to see from (3a) that H' cannot coincide with H, unless S is a quadric.

The hyperboloids H and H' have the straight line g' in common. The rest of their intersection is therefore, in general, a space cubic. We shall call it the *derivative cubic*, and discuss some of its properties. It is interesting to notice that *we obtain in this way associated with every ruled surface, surfaces containing a single infinity of twisted cubics.* We shall consider some of the properties of these surfaces.

Let us again assume $\Theta_4 \neq 0$ and let P_y and P_z be the flecnodes of g. Then, it follows from (1) and (3a) that we may take

$$(4)\qquad \begin{aligned} x_1 &= t x_2, \qquad x_2 = J(u_{11} - u_{22})t, \qquad x_3 = t x_4, \\ x_4 &= -\lambda_{12} u_{22}^2 t^2 + u_{11} u_{22} (\lambda_{11} - \lambda_{22}) t + \lambda_{21} u_{11}^2 \end{aligned}$$

as the parametric equations of the cubic. From these, the following corollaries follow at once. *If* $\lambda_{11} - \lambda_{22} = 0$, *the derivative cubic, and therefore the hyperboloid* H', *intersects* g *in two points which are harmonic conjugates with respect to the flecnodes. If*

$$(\lambda_{11} - \lambda_{22})^2 + 4\lambda_{12}\lambda_{21} = 0,$$

the cubic is tangent to g. *The congruence* Γ *contains* ∞^2 *surfaces* S' *corresponding to each of these properties of the derivative cubic.* For, the corresponding equations for η are again of the second order.

The equation of a plane which is tangent to H' at a point (x_1', x_2', x_3', x_4'), P_y and P_z being flecnodes, is

$$(5)\qquad \begin{aligned} &[-2\lambda_{12} u_{22}^2 x_1' + u_{11} u_{22} (\lambda_{11} - \lambda_{22}) x_2' + u_{11} u_{22}^2 x_4'] x_1 \\ &\quad + [u_{11} u_{22} (\lambda_{11} - \lambda_{22}) x_1' + 2\lambda_{21} u_{11}^2 x_2' - u_{11}^2 u_{22} x_3'] x_2 \\ &\quad - u_{11}^2 u_{22} x_2' x_3 + u_{11} u_{22}^2 x_1' x_4 = 0. \end{aligned}$$

Consider the two points of g'' which correspond to P_y and P_z, viz.,

$$P_1'' = (u_{11}, 0, \lambda_{11}, \lambda_{12}), \quad P_2'' = (0, u_{22}, \lambda_{21}, \lambda_{22}).$$

The equations of the two planes tangent to S'' or H' at these points respectively, are

$$\begin{aligned} -u_{22}\lambda_{12} x_1 - u_{11}\lambda_{22} x_2 + u_{11} u_{22} x_4 &= 0, \\ u_{22}\lambda_{11} x_1 + u_{11}\lambda_{21} x_2 - u_{11} u_{22} x_3 &= 0. \end{aligned}$$

They intersect g, i. e., the line $x_3 = x_4 = 0$, in two points

$$P_1 = (u_{11}\lambda_{22}, -u_{22}\lambda_{12}, 0, 0), \quad P_2 = (u_{11}\lambda_{21}, -u_{22}\lambda_{11}, 0, 0),$$

which coincide, if and only if $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0$, i e., if and only if g'' intersects g, as is moreover geometrically evident. P_1 and P_2 are harmonic conjugates with respect to the flecnodes if $\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21} = 0$, i. e., *under this condition the planes, tangent to the second derivative at the points which correspond to the flecnodes of* g, *divide these flecnodes harmonically.*

The line $P_z P_\varrho$ has, besides P_ϱ, another point in common with H'. Its coordinates are found to be $(0, u_{22}, \lambda_{21}, 0)$. Similarly $P_y P_\sigma$ intersects H' in a second point $(u_{11}, 0, 0, \lambda_{12})$. Join these two points. The coordinates of any point on the line joining them will be $(\mu u_{11}, \lambda u_{22}, \lambda\lambda_{21}, \mu\lambda_{12})$, where $\lambda : \mu$ determines the position of the particular point It is easily seen, by substituting in (3a), that this line is entirely on H' if $\lambda_{11} - \lambda_{22} = 0$, and in no other case, provided

that S' is not developable. We have therefore the following result. *Corresponding to every point of S we have a point of S'. If each of the two flecnodes on a generator g of S be joined to the point of S' which corresponds to the other, the two straight lines thus obtained intersect the hyperboloid H' osculating S' along g' in two new points. The line joining these latter points lies entirely on H', if H' intersects g in two points which are harmonic conjugates with respect to the flecnodes. The converse of the theorem is also true.*

Let us introduce the following abbreviations:

$$
(6) \qquad \begin{aligned}
&A = 2u_{11}u_{22}(u_{11} - u_{22}), \quad \psi = Bt_1^2 + 2Ct_1t_2 + Dt_2^2, \\
&B = v_{12}u_{22}, \quad C = \frac{1}{2}(u_{11}v_{22} - u_{22}v_{11}), \quad D = -v_{21}u_{11},
\end{aligned}
$$

and let us write the parameter t of the cubic curve in homogeneous form. Then we may write instead of (4),

$$
(7) \qquad \begin{aligned}
x_1 &= At_1^2t_2, \quad x_3 = (Bt_1^2 + 2Ct_1t_2 + Dt_2^2)t_1 = \psi t_1, \\
x_2 &= At_1t_2^2, \quad x_4 = (Bt_1^2 + 2Ct_1t_2 + Dt_2^2)t_2 = \psi t_2.
\end{aligned}
$$

If the cubic degenerates, each irreducible part will be a plane curve (a conic or a straight line). If, therefore, the cubic degenerates, it must be possible to satisfy the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

for all values of $t_1:t_2$, the coefficients being independent of $t_1:t_2$. If we substitute in this equation the values (7), and equate to zero the coefficients of t_1^3, $t_1^2t_2$, etc., we find

$$
(8) \quad a_3B = 0, \quad a_4D = 0, \quad a_1A + 2a_3C + a_4B = 0, \quad a_2A + a_3D + 2a_4C = 0.
$$

Let us assume first that neither B nor D vanishes. Then $a_3 = a_4 = 0$, and $A = 0$; for, if A were not zero, we would have also $a_1 = a_2 = 0$, i. e., there would be no plane containing the (supposedly) degenerate cubic. But from $A = 0$ follows either $u_{11} - u_{22} = 0$, which would make S a quadric, or else u_{11} or u_{22} would vanish, which however contradicts the assumption that B and D shall not be zero.

Let us now assume $B \neq 0$, $D = 0$. Then $a_3 = 0$, and either u_{11} or v_{21} must vanish, i. e., either S' is developable or S has a straight line directrix. Similarly if $B = 0$, $D \neq 0$. Finally if $B = 0$, $D = 0$, either S has two straight line directrices, or else it has one while S' is developable We have, therefore, the following theorem. *If a ruled surface, with distinct branches to its flecnode curve, has one or more straight line directrices the derivative cubic always degenerates. In all other cases, the only way to obtain a degenerate derivative cubic consists in taking as derivative ruled surface of S, one of the developables of the congruence Γ.*

Another question at once suggests itself. To every value of x, i. e., to every generator of S there belongs a derivative cubic. In general, the cubics belonging to values of x, differing from each other by an infinitesimal δx, will not intersect. Their shortest distance will be an infinitesimal of the same order as δx. It may happen however that, for an appropriately chosen variable, this distance becomes infinitesimal of a higher order, or as we may say briefly, that consecutive cubics intersect. We ask now: is it possible to choose the independent variable in such a way that every pair of consecutive derivative cubics may intersect?

By putting $y = y_k$, $z = z_k$ $(k = 1, 2, 3, 4)$ in

$$\Phi = A t_1 t_2 (t_1 y + t_2 z) + \psi (t_1 \varrho + t_2 \sigma),$$

we obtain the coordinates of any point P_Φ on the cubic. As x changes we go from one cubic to another; as $t_1 : t_2$ changes we go from one point on a certain cubic to another point of the same curve. Equation (9) gives therefore, if both x and $t_1 : t_2$ be taken as variables, the locus of all such points P_Φ, i. e., the surface generated by all of the derivative cubics of S. If $t_1 : t_2$ be chosen as a function of x, a curve is picked out upon this surface. Let us differentiate Φ totally, i. e., assuming that t_1 and t_2 are functions of x, and consider the quantity

$$\Phi + \frac{d\Phi}{dx}\delta x,$$

where δx is an infinitesimal. This will clearly represent the coordinates of a point on the adjacent derivative cubic determined by the parameters $x + \delta x$ and $t_k + dt_k/dx\,\delta x$. If the original cubic and this second one, infinitesimally close to it, intersect, it must be possible to choose t_k as functions of x in such a way, that the corresponding points of the two curves shall coincide up to infinitesimals of higher than the first order. Therefore $d\Phi/dx$ must differ from a multiple of Φ only by an infinitesimal quantity. Proceeding to the limit we must therefore have

$$\frac{d\Phi}{dx} = \omega \Phi. \tag{10}$$

We find by differentiation

$$\begin{aligned} \frac{d\Phi}{dx} = {} & [(t_1 t_2' + t_1' t_2) A + A' t_1 t_2](t_1 y + t_2 z) \\ & + y\left[A t_1 t_2 \left(t_1' - \frac{1}{2} p_{11} t_1 - \frac{1}{2} p_{21} t_2\right) + \frac{1}{2}\psi u_{11} t_1\right] \\ & + z\left[A t_1 t_2 \left(t_2' - \frac{1}{2} p_{12} t_1 - \frac{1}{2} p_{22} t_2\right) + \frac{1}{2}\psi u_{22} t_2\right] \\ & + \varrho\left[\frac{1}{2} A t_1^2 t_2 + \psi\left(t_1' - \frac{1}{2} p_{11} t_1 - \frac{1}{2} p_{21} t_2\right) + \psi' t_1\right] \\ & + \sigma\left[\frac{1}{2} A t_1 t_2^2 + \psi\left(t_2' - \frac{1}{2} p_{12} t_1 - \frac{1}{2} p_{22} t_2\right) + \psi' t_2\right], \end{aligned} \tag{10a}$$

denoting as usual differentiation by strokes.

If we substitute in (10) we find the following four equations:

$$
(11)\quad
\begin{aligned}
&\text{(a)}\ [(t_1 t_2' + t_1' t_2) A + A' t_1 t_2] t_1 + A t_1 t_2 \left(t_1' - \tfrac{1}{2} p_{11} t_1 - \tfrac{1}{2} p_{21} t_2\right) \\
&\qquad\qquad + \tfrac{1}{2} \psi u_{11} t_1 = \omega A t_1^2 t_2, \\
&\text{(b)}\ [(t_1 t_2' + t_1' t_2) A + A' t_1 t_2] t_2 + A t_1 t_2 \left(t_2' - \tfrac{1}{2} p_{12} t_1 - \tfrac{1}{2} p_{22} t_2\right) \\
&\qquad\qquad + \tfrac{1}{2} \psi u_{22} t_2 = \omega A t_1 t_2^2, \\
&\text{(c)}\ \tfrac{1}{2} A t_1^2 t_2 + \psi \left(t_1' - \tfrac{1}{2} p_{11} t_1 - \tfrac{1}{2} p_{21} t_2\right) + \psi' t_1 = \omega \psi t_1, \\
&\text{(d)}\ \tfrac{1}{2} A t_1 t_2^2 + \psi \left(t_2' - \tfrac{1}{2} p_{12} t_1 - \tfrac{1}{2} p_{22} t_2\right) + \psi' t_2 = \omega \psi t_2.
\end{aligned}
$$

If we multiply both members of (a) by t_2, of (b) by $-t_1$, and add, and if we treat (c) and (d) in the same way, we find

$$
\begin{aligned}
&\text{(e)}\ A t_1 t_2^2 \left(t_1' - \tfrac{1}{2} p_{11} t_1 - \tfrac{1}{2} p_{21} t_2\right) - A t_1^2 t_2 \left(t_2' - \tfrac{1}{2} p_{12} t_1 - \tfrac{1}{2} p_{22} t_2\right) \\
&\qquad\qquad + \tfrac{1}{2} \psi t_1 t_2 (u_{11} - u_{22}) = 0, \\
&\text{(f)}\ \psi t_2 \left(t_1' - \tfrac{1}{2} p_{11} t_1 - \tfrac{1}{2} p_{21} t_2\right) - \psi t_1 \left(t_2' - \tfrac{1}{2} p_{12} t_1 - \tfrac{1}{2} p_{22} t_2\right) = 0.
\end{aligned}
$$

Let us assume first $\psi \neq 0$. Divide both members of (f) by ψ, and compare with (e). We find

$$\psi t_1 t_2 (u_{11} - u_{22}) = 0,$$

i. e., either S is a quadric, or else either t_1 or t_2 must vanish identically. Assume $t_1 = 0$. Then (e) is satisfied. From (f) we should find $p_{21} = 0$, for t_2 cannot be zero, since the ratio $t_1 : t_2$ is the parameter which determines a point on the cubic. But from $u_{21} = p_{21} = 0$ follows further $v_{21} = 0$, i. e, $D = 0$ so that in this case $\psi = B t_1^2 + 2 C t_1 t_2 = 0$ contrary to our assumption. It is, therefore, impossible to satisfy (11) except by putting $\psi = 0$. According to (c) and (d) this gives either $t_1 = 0$, or $t_2 = 0$, or $A = 0$. Assume $A \neq 0$ and say $t_1 = 0$. All equations (11) are now satisfied. But from $t_1 = 0$, $\psi = 0$ follows $D = 0$, i. e., either u_{11} or v_{21} must vanish, i. e., either S has a straight line directrix or else S' is developable. If however $A = 0$, either S is a quadric or else S' is developable.

In connection with our last theorem, we may therefore say: *It is impossible to choose the surface S' of the congruence Γ in such a way that consecutive derivative cubics may intersect, except in the trivial cases when the cubics are degenerate.*

If in (9) we put $\psi = 0$, we obtain the locus of the intersections of the cubic with the generator of S to which it belongs, i e., a

certain curve cutting every generator twice. This may be an asymptotic curve. It is, in fact, if the further conditions

$$(12)\qquad 2t_1' - p_{11}t_1 - p_{21}t_2 = \omega t_1, \quad 2t_2' - p_{12}t_1 - p_{22}t_2 = \omega t_2$$

are satisfied, where ω is arbitrary. For, as (10a) shows, the line joining P_Φ to $P_{\Phi + \Phi'\delta x}$, i. e., the tangent to this curve, is then a generator of the second kind on the hyperboloid osculating S along g. In other words it is a tangent to an asymptotic curve of S

In general, of course, the conditions (12) and $\psi = 0$ can not both be satisfied at once. The question is: when are these conditions consistent? We find from (12)

$$t_1' = \frac{1}{2}\,[(p_{11} + \omega)t_1 + p_{21}t_2], \quad t_2' = \frac{1}{2}\,[p_{12}t_1 + (p_{22} + \omega)t_2].$$

Let us substitute these values of t_1' and t_2' in the equation $d\psi/dx = 0$. We shall find

$$\begin{aligned}[B' + B(p_{11} + \omega) + Cp_{12}]t_1^2 + [D' + D(p_{22} + \omega) + Cp_{21}]t_2^2 \\ + [2C' + Bp_{12} + Dp_{21} + C(p_{11} + p_{22} + 2\omega)]t_1t_2 = 0,\end{aligned}$$

which compared with $\psi = 0$ gives,

$$\begin{aligned}B' + B(p_{11} + \omega) + Cp_{12} &= \tau B,\\ D' + D(p_{22} + \omega) + Cp_{21} &= \tau D,\\ 2C' + Bp_{12} + Dp_{21} + C(p_{11} + p_{22} + 2\omega) &= 2\tau C,\end{aligned}$$

where τ is a proportionality factor. Eliminating τ, we find

$$(13)\quad \begin{aligned}&\text{(a)}\ 2(B'C - BC') + BC(p_{11} - p_{22}) + (2C^2 - BD)p_{12} - B^2p_{21} = 0,\\ &\text{(b)}\ 2(D'C - DC') + DC(p_{22} - p_{11}) + (2C^2 - BD)p_{21} - D^2p_{12} = 0,\\ &\text{(c)}\ BD' - B'D + BD(p_{11} - p_{22}) + C(Dp_{12} - Bp_{21}) = 0.\end{aligned}$$

We may always assume that our system of differential equations has been so written that $p_{11} - p_{22} = 0$. Equations (13) may then be regarded as three homogeneous linear equations for $1, p_{12}, p_{21}$. Their determinant must therefore be zero. This gives rise to three alternatives: either $C^2 - BD$ or C or $BD' - B'D$ must vanish.

Consider first the case $C^2 - BD = 0$. Equations (13) become

$$(14)\quad \begin{aligned}2(B'C - BC') - B^2p_{21} = 0, \quad 2(D'C - DC') - D^2p_{12} = 0,\\ BD' - B'D + CDp_{12} - CBp_{21} = 0.\end{aligned}$$

If we multiply both members of the first two equations by CD and $-CB$ respectively, add and make use of $C^2 = BD$, we find

$$2C^2(DB' - D'B) + C^2(CDp_{12} - CBp_{21}) = 0,$$

whence, if $C \neq 0$,

$$-2(BD' - B'D) + CDp_{12} - CBp_{21} = 0.$$

But if we consider the last equation of (14), this gives $BD' - B'D = 0$, $Dp_{12} - Bp_{21} = 0$, which last equation may be written

$$p_{12}p_{21}(u_{11} - u_{22})^2 = 0,$$

i. e., the surface S has at least one straight line directrix.

In the second place let $C = 0$. Then (13) becomes

$$B(Dp_{12} + Bp_{21}) = 0, \quad D(Dp_{12} + Bp_{21}) = 0, \quad BD' - B'D = 0.$$

According to the first two equations S must either have at least two straight line directrices, or else $u_{11} + u_{22}$ must be zero. The third condition gives, on substituting the values of B and D,

$$\frac{u'_{22}}{u_{22}} - \frac{u'_{11}}{u_{11}} + \frac{v'_{12}}{v_{12}} - \frac{v'_{21}}{v_{21}} = 0,$$

if we assume that S has no straight line directrix, so that v_{12} and v_{21} are not zero. By integration we find

$$\frac{v_{12}}{v_{21}}\frac{u_{22}}{u_{11}} = \text{const.},$$

whence, since $u_{11} + u_{22} = 0$ in this case,

$$\frac{p_{12}}{p_{21}} = \text{const.}$$

But if we substitute into the condition $\varDelta = 0$ for a surface belonging to a linear complex the assumptions $u_{12} = u_{21} = p_{11} - p_{22} = 0$, which we have made, we find either $u_{11} - u_{22} = 0$, i. e., S is a quadric, or $p_{12}p'_{21} - p'_{12}p_{21} = 0$, i. e., $p_{12}/p_{21} = \text{const}$ Therefore we see that if $C = 0$, S either has straight line directrices, or at least belongs to a linear complex. Moreover, if S has no straight line directrices, the independent variable must be so chosen as to make $u_{11} + u_{22}$ vanish.

If finally $BD' - B'D = 0$, we find from (13) either $C = 0$, which leads us back to the case just considered, or else $Dp_{12} - Bp_{21}$ must vanish, which gives again a surface S with a least one straight line directrix.

In all of these cases the surface S belongs to a linear complex. If we leave aside the trivial cases, we may say that *if a ruled surface belongs to a linear complex, and if the independent variable is so chosen that $u_{11} + u_{22} = 0$, the surface of derivative cubics determines an asymptotic curve upon it which intersects every generator twice.*

We have seen in Chapter IX, that there exists in the congruence Γ a single infinity of ruled surfaces S' for which $u_{11} + u_{22} = 0$. They are those ruled surfaces of Γ, whose intersections with the flecnode surface of S are asymptotic lines upon them. *But we can also characterize them by saying that such a surface is made up of the lines of Γ which intersect any asymptotic curve on the flecnode surface*

of S. In fact, if we assume $u_{12} = u_{21} = 0$, $p_{11} = p_{22} = 0$, we may write the equations of the sheet F' of the flecnode surface as follows [cf. Chapter VI, equ. (15)]:

$$
\begin{aligned}
&y'' - 2\frac{q_{12}}{p_{12}} y' - \varrho' - q_{11} y + \frac{q_{12}}{p_{12}} \varrho = 0, \\
(15) \qquad &\varrho'' + [2(q_{11} + q_{22}) - p_{12} p_{21}] y' - 2\frac{q_{12}}{p_{12}} \varrho' \\
&\quad + \left[2 q'_{11} - p_{12} q_{21} - 4\frac{q_{12}}{p_{12}} q_{11}\right] y - q_{22} \varrho = 0.
\end{aligned}
$$

If $u_{11} + u_{22} = 0$, the coefficient of y' in the second equation vanishes, which proves that the curve C_ϱ is an asymptotic curve on F'. But from equations (16) of Chapter VI we see that C_σ is then also an asymptotic curve on F''.

The asymptotic curves on F' and F'' must, therefore, correspond to each other. The congruence Γ is a so-called W-congruence.[1])

We may now state our theorem as follows: *In order that the surface of derivative cubics may intersect the ruled surface S in an asymptotic curve, S must belong to a linear complex. Moreover, its derivative ruled surface must intersect the flecnode surface of S along an asymptotic curve.*

The asymptotic curve on S, thus determined, is unique. For the ratio $B:D$, which determines it, cannot be changed by any transformation of the variables which preserves the conditions

$$p_{11} - p_{22} = u_{12} = u_{21} = u_{11} + u_{22} = 0.$$

We may, therefore, take any asymptotic curve of the flecnode surface and consider the ruled surface of Γ made up of the lines which intersect it. We obtain thus as a consequence a single infinity of surfaces made up of derivative cubics. All of of these intersect S along the same asymptotic curve.

But we notice further that $C = 0$. Therefore, *this asymptotic curve intersects every generator in two points which are harmonic conjugates with respect to the flecnodes* We shall meet this special asymptotic curve again in a later paragraph.

Another question suggests itself. Is it possible to choose the independent variable of our system in such a way that the derivative cubics shall be asymptotic lines on the surface generated by them?

In order to answer this question, we must first find the coordinates of the osculating plane of the cubic at any one of its points. For the moment we prefer to take the equation of the cubic referred to a non-homogeneous parameter t, i. e., in the form

$$x_1 = At^2, \quad x_2 = At, \quad x_3 = Bt^3 + 2Ct^2 + Dt, \quad x_4 = Bt^2 + 2Ct + D.$$

1) Cf. *Bianchi-Lukat, Vorlesungen über Differentialgeometrie,* p. 315, for the general theory of W-congruences.

Its intersections with the plane $\Sigma u_i x_i = 0$, whose coordinates are $(u_1, \ldots u_4)$, will be given by solving the cubic equation

$$Bu_3 t^3 + (Au_1 + 2Cu_3 + Bu_4)t^2 + (Au_2 + 2Cu_4 + Du_3)t + Du_4 = 0.$$

The three roots of this cubic must coincide if the plane is an osculating plane of the cubic curve. They must, therefore, also satisfy the equations obtained from the above by twofold differentiation with respect to t. This gives the following conditions:

$$\begin{aligned} &3Bu_3 t + Au_1 + 2Cu_3 + Bu_4 = 0, \\ &(Au_1 + 2Cu_3 + Bu_4)t + Au_2 + 2Cu_4 + Du_3 = 0, \\ &(Au_2 + 2Cu_4 + Du_3)t + 3Du_4 = 0. \end{aligned}$$

Of course, only the ratios of $u_1 \ldots u_4$ are of interest, so that we may multiply $u_1 \ldots u_4$ by a common factor if we please. We find

$$(16) \qquad \begin{aligned} u_1 &= B^2 t^3 - 3BDt - 2CD, \quad & u_3 &= AD, \\ u_2 &= 2BCt^3 + 3BDt^2 - D^2, \quad & u_4 &= -ABt^3, \end{aligned}$$

as the coordinates of the plane osculating the derivative cubic belonging to the generator g at the point whose parameter is t, or in homogeneous form,

$$(17) \qquad \begin{aligned} u_1 &= B^2 t_1^3 - 3BDt_1 t_2^2 - 2CDt_2^3, \quad & u_3 &= ADt_2^3, \\ u_2 &= 2BCt_1^3 + 3BDt_1^2 t_2 - D^2 t_2^3, \quad & u_4 &= -ABt_1^3. \end{aligned}$$

We have already made use of the expression

$$\Phi = At_1 t_2 (t_1 y + t_2 z) + \psi (t_1 \varrho + t_2 \sigma)$$

as giving the coordinates of a point $(t_1 : t_2)$ of the cubic curve which belongs to the argument x, or as giving the coordinates of a point $(x, t_1 : t_2)$ of the surface formed by the aggregate of all of these curves. The plane which is tangent to this surface at the point $(x, t_1 : t_2)$ must contain also the point $t_1 : t_2$ of the adjacent cubic, i. e, the point whose coordinates are given by

$$\Phi + \frac{\partial \Phi}{\partial x} \delta x,$$

where, in forming $\partial \Phi / \partial x$, x, t_1 and t_2 are regarded as independent variables. The tangent plane must therefore contain the point whose coordinates are $\partial \Phi / \partial x$. We have

$$\begin{aligned} \frac{\partial \Phi}{\partial x} = {} & y \left[A' t_1^2 t_2 - \frac{1}{2} At_1 t_2 (p_{11} t_1 + p_{21} t_2) + \frac{1}{2} \psi u_{11} t_1 \right] \\ & + z \left[A' t_1 t_2^2 - \frac{1}{2} At_1 t_2 (p_{12} t_1 + p_{22} t_2) + \frac{1}{2} \psi u_{22} t_2 \right] \\ & + \varrho \left[\frac{1}{2} At_1^2 t_2 - \frac{1}{2} \psi (p_{11} t_1 + p_{21} t_2) + \frac{\partial \psi}{\partial x} t_1 \right] \\ & + \sigma \left[\frac{1}{2} At_1 t_2^2 - \frac{1}{2} \psi (p_{12} t_1 + p_{22} t_2) + \frac{\partial \psi}{\partial x} t_2 \right]. \end{aligned}$$

The point, whose coordinates $(\xi_1, \ldots \xi_4)$ are the coefficients of y, z, ρ, σ in this expression, is in the tangent plane of the point $(x, t_1 : t_2)$. If, therefore, the cubic curve is an asymptotic line upon the surface, its osculating plane must contain the point $(\xi_1, \ldots \xi_4)$, i. e., we must have

$$u_1\xi_1 + u_2\xi_2 + u_3\xi_3 + u_4\xi_4 = 0. \tag{18}$$

We find

$$\begin{aligned}
&\xi_1 = u_{11}Bt_1^3 + (2A' + 2u_{11}C - Ap_{11})t_1^2t_2 + (u_{11}D - Ap_{21})t_1t_2^2,\\
&\xi_2 = u_{22}Dt_2^3 + (u_{22}B - Ap_{12})t_1^2t_2 + (2A' + 2u_{22}C - Ap_{22})t_1t_2^2,\\
&\xi_3 = (2B' - Bp_{11})t_1^3 + (A - Bp_{21} + 4C'' - 2Cp_{11})t_1^2t_2\\
&\qquad + (2D' - Dp_{11} - 2Cp_{21})t_1t_2^2 - Dp_{21}t_2^3\\
&\xi_4 = -Bp_{12}t_1^3 + (2B' - Bp_{22} - 2Cp_{12})t_1^2t_2\\
&\qquad + (A - Dp_{12} + 4C'' - 2Cp_{22})t_1t_2^2 + (2D' - Dp_{22})t_2^3.
\end{aligned} \tag{19}$$

If these values, and the values (17) for $u_1, \ldots, u_4$ be substituted in (18), and the coefficients of t_1^6, $t_1^5t_2$, etc., be successively equated to zero, the following seven equations make their appearance:

$$\begin{aligned}
&u_{11}B^3 + AB^2p_{12} = 0,\\
&B^2(2A' + 2u_{11}C - Ap_{11}) + 2BC(u_{22}B - Ap_{12})\\
&\qquad\qquad - AB(2B' - Bp_{22} - 2Cp_{12}) = 0,\\
&B^2(u_{11}D - Ap_{21}) - 3B^2Du_{11} + 2BC(2A' + 2u_{22}C - Ap_{22})\\
&\qquad\qquad + 3BD(u_{22}B - Ap_{12}) - AB(A - Dp_{12} + 4C' - 2Cp_{22}) = 0,\\
&-3BD(2A' + 2u_{11}C - Ap_{11}) - 2BCD(u_{11} - u_{22})\\
&\qquad\qquad + 3BD(2A' + 2u_{22}C - Ap_{22})\\
&\qquad\qquad + AD(2B' - Bp_{11}) - AB(2D' - Dp_{22}) = 0,\\
&-3BD(u_{11}D - Ap_{21}) - 2CD(2A' + 2u_{11}C - Ap_{11})\\
&\qquad\qquad + 3BD^2u_{22} - D^2(u_{22}B - Ap_{12})\\
&\qquad\qquad + AD(A - Bp_{21} + 4C' - 2Cp_{11}) = 0,\\
&-2CD(u_{11}D - Ap_{21}) - D^2(2A' + 2u_{22}C - Ap_{22})\\
&\qquad\qquad + AD(2D' - Dp_{11} - 2Cp_{21}) = 0,\\
&-u_{22}D^3 - AD^2p_{21} = 0.
\end{aligned} \tag{20}$$

Let us assume that both B and D are different from zero. Then the first and last equations give

$$u_{11}u_{22}v_{12} + 2u_{11}u_{22}(u_{11} - u_{22})p_{12} = 0,$$
$$u_{11}u_{22}v_{21} - 2u_{11}u_{22}(u_{11} - u_{22})p_{21} = 0,$$

or

$$u_{11}u_{22}(u_{11} - u_{22})p_{12} = u_{11}u_{22}(u_{11} - u_{22})p_{21} = 0.$$

But all of the possibilities here suggested give zero values to either B or D, or both, except $u_{11} - u_{22} = 0$, in which case S is a quadric. But this is only apparently an exception, arising from the fact that in this case the flecnode curve is indeterminate. If two of the straight lines of the second set be taken for the curves C_y and C_z, we have in this case also $B = D = 0$.

There remain two possibilities. If $B = D = 0$ all of the equations (20) are satisfied. If S' is not developable, S must belong to a linear congruence. If S' is developable, it is sufficient for S to have one straight line directrix. In either case the derivative cubic degenerates into a straight line, and is therefore obviously an asymptotic curve upon the surface of cubics.

Finally, it might happen that one only of the two quantities B and D is zero. Say $B = 0$, $D \neq 0$. Then the first four equations of (20) are satisfied. The other three become

$$\begin{aligned}&- 4u_{11}C^2 + ADp_{12} + A^2 = 0,\\ &- 2Cu_{11}D - D(2A' + 2u_{22}C - Ap_{22}) + A(2D' - Dp_{11}) = 0,\\ &- u_{22}D - Ap_{21} = 0.\end{aligned}$$

From the last of these we find

$$u_{11}u_{22}(u_{11} - u_{22})p_{21} = 0,$$

whence, since $D \neq 0$, follows $u_{22} = 0$, and therefore $A = 0$. The second equation now gives us $C = 0$, which also satisfies the first We have again S' a developable surface, so that the cubic degenerates. Therefore

The derivative cubics of a ruled surface are asymptotic curves upon the surface formed by their totality, only in the trivial cases when they degenerate into straight lines.

§ 2. Null-system of the derivative cubic.

A twisted cubic always determines a null-system, i. e., a point-to-plane correspondence with incident elements. Geometrically this correspondence may be constructed as follows. An arbitrary plane intersects the curve in three points. The three planes, which osculate the curve in these points, intersect again in a point which is situated in the original plane. This is the point which corresponds to the plane in the null-system of the cubic.

We shall now set up the equations for this null-system. For this purpose it is more convenient to use the equations of the cubic referred to a non-homogeneous parameter t.

Let t_1, t_2, t_3 be the three values of t which correspond to the three points in which the plane

$$v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = 0$$

intersects the cubic. Then t_1, t_2, t_3 are the roots of the cubic equation

$$Bv_3t^3 + (Av_1 + 2Cv_3 + Bv_4)t^2 + (Av_2 + Dv_3 + 2Cv_4)t + Dv_4 = 0.$$

Therefore we shall have

$$\frac{Av_1 + 2Cv_3 + Bv_4}{Bv_3} = -(t_1 + t_2 + t_3),$$

$$\frac{Av_2 + Dv_3 + 2Cv_4}{Bv_3} = t_2t_3 + t_3t_1 + t_1t_2,$$

$$\frac{Dv_4}{Bv_3} = -t_1t_2t_3.$$

If we solve these equations for v_1/v_3, v_2/v_3, v_4/v_3, then make them homogeneous, and multiply $v_1 \ldots v_4$ by the common factor A, we shall find

$$(21)\quad \begin{aligned} v_1 &= -2CD + B^2t_1t_2t_3 - BD(t_1+t_2+t_3), & v_3 &= AD, \\ v_2 &= -D^2 + 2BCt_1t_2t_3 + BD(t_2t_3 + t_3t_1 + t_1t_2), & v_4 &= -ABt_1t_2t_3. \end{aligned}$$

The coordinates of the planes which osculate the curve in the three points t_1, t_2, t_3, are, according to (16),

$$\begin{aligned} u_1^{(k)} &= B^2t_k^3 - 3BDt_k - 2CD, & u_3^{(k)} &= AD \\ u_2^{(k)} &= 2BCt_k^3 + 3BDt_k^2 - D^2, & u_4^{(k)} &= -ABt_k^3 \end{aligned} \quad (k = 1, 2, 3).$$

Let $x_1, \ldots x_4$ be the coordinates of the point of intersection of these three planes. We must then have

$$\sum_{i=1}^{4} u_i^{(k)} x_i = 0 \quad (k = 1, 2, 3).$$

Solving these equations we find

$$(22)\quad \begin{aligned} x_1 &= A(t_2t_3 + t_3t_1 + t_1t_2), \\ x_2 &= A(t_1 + t_2 + t_3), \\ x_3 &= 3Bt_1t_2t_3 + 2C(t_2t_3 + t_3t_1 + t_1t_2) + D(t_1 + t_2 + t_3), \\ x_4 &= B(t_2t_3 + t_3t_1 + t_1t_2) + 2C(t_1 + t_2 + t_3) + 3D, \end{aligned}$$

and a simple calculation will show that $\Sigma v_i x_i = 0$, i. e., as we have stated, the point of intersection of the three osculating planes lies in the plane of the three points of osculation.

In our null-system then, the plane (21) and the point (22) correspond to each other. To find the explicit equations for this correspondence, we need only eliminate t_1, t_2, t_3 between equations (21) and (22). Denoting by ω and ω' two proportionality factors, we find:

$$
(23)\quad
\begin{aligned}
\omega v_1 &= \quad * \quad + 4(C^2 - BD)x_2 + ABx_3 - 2ACx_4,\\
\omega v_2 &= -4(C^2 - BD)x_1 + \quad * \quad + 2ACx_3 - ADx_4,\\
\omega v_3 &= -ABx_1 - 2ACx_2 + \quad * \quad + A^2x_4,\\
\omega v_4 &= 2ACx_1 + ADx_2 - A^2x_3 + \quad *;
\end{aligned}
$$

and

$$
(24)\quad
\begin{aligned}
\omega' x_1 &= \quad * \quad + A^2v_2 + ADv_3 + 2ACv_4,\\
\omega' x_2 &= -A^2v_1 + \quad * \quad - 2ACv_3 - ABv_4,\\
\omega' x_3 &= -ADv_1 + 2ACv_2 + \quad * \quad + 4(C^2 - BD)v_4,\\
\omega' x_4 &= -2ACv_1 + ABv_2 - 4(C^2 - BD)v_3 + \quad * .
\end{aligned}
$$

But, associated with the null-system, we have a linear complex, made up of all of the lines which pass through a point and lie at the same time in the plane corresponding to this point in the null-system. Introduce line coordinates by putting

$$\omega_{ik} = x_i y_k - x_k y_i, \quad \omega'_{ik} = v_i w_k - v_k w_i,$$

where $x_1 \dots x_4$ and $y_1 \dots y_4$ are the coordinates of two points on the line, and where $v_1 \dots v_4$, $w_1 \dots w_4$ are the coordinates of two planes containing the line. *Then the equation of the complex may be written in either of the forms*

$$
(25)\quad
\begin{aligned}
-4(C^2 - BD)\omega_{12} - AB\omega_{13} + 2AC\omega_{14}&\\
-2AC\omega_{23} - AD\omega_{42} - A^2\omega_{34} &= 0,
\end{aligned}
$$

$$
(26)\quad
\begin{aligned}
-4(C^2 - BD)\omega'_{34} - AB\omega'_{42} + 2AC\omega'_{23}&\\
-2AC\omega'_{14} - AD\omega'_{13} - A^2\omega'_{12} &= 0.
\end{aligned}
$$

This complex becomes special if $A^2BD = 0$, i. e., only if the cubic degenerates.

To the flecnode $P_y (x_2 = x_3 = x_4 = 0)$ corresponds the plane

$$[0, -4(C^2 - BD), -AB, 2AC].$$

Therefore, if $C^2 - BD = 0$, i. e., *if the derivative cubic is tangent to the generator g, the corresponding plane passes through g.* If $C = 0$, i. e, $u_{11}/u_{22} = \text{const.}$ the plane passes through $P_\sigma P_y$. Therefore, *if the intersections of the cubic with g and the flecnodes form a harmonic group on g, the plane corresponding to each flecnode passes through that point of the derived ruled surface which corresponds to the other.*

§ 3. The osculating linear complex.

There is another linear complex associated with every generator of a ruled surface, even more important than the one just considered. A linear complex is determined by five of its lines, provided that these have no two straight line intersectors. Let us consider five

generators of a ruled surface, g and four others. As these four generators approach coincidence with g, a definite linear complex will in general be obtained as a limit. We shall speak of it as the linear complex osculating S along g.

Instead of determining the complex by five consecutive generators of S, it will be advisable to determine it by means of two pairs of lines which are reciprocal polars with respect to it. Two such pairs are obviously constituted by the flecnode tangent of g and of another generator infinitesimally close to g Let us denote by g_x, f_x', f_x'', g_x' the generator of S, the first and the second flecnode tangents and the generator of S', which belong to the argument x. Similarly we denote by $g_{x+\delta x}$, etc., the corresponding lines belonging to the argument $x + \delta x$, where δx is an infinitesimal.

Clearly f_x' and f_x'' are the directrices of the osculating linear congruence, (determined by four consecutive generators). Therefore, all lines intersecting f_x' and f_x'' belong to the osculating linear complex, whose equation must, therefore, be of the form

$$a\omega_{13} + b\omega_{42} = 0. \tag{27}$$

Change the parameter x by an infinitesimal quantity δx. The flecnode tangents $f'_{x+\delta x}$ and $f''_{x+\delta x}$ must again be the directrices of a linear congruence contained in the complex. We have

$$y_{x+\delta x} = y_x + y_x'\delta x, \quad \varrho_{x+\delta x} = \varrho_x + \varrho_x'\delta x, \text{ etc}$$

If we substitute the values of y', ϱ', etc. from Chapter IV equations (112) and Chapter IX equations (2), we shall find

$$2y_{x+\delta x} = 2y + (\varrho - p_{11}y - p_{12}z)\delta x,$$

$$2z_{x+\delta x} = 2z + (\sigma - p_{21}y - p_{22}z)\delta x,$$

$$2\varrho_{x+\delta x} = 2\varrho + (u_{11}y + u_{12}z - p_{11}\varrho - p_{12}\sigma)\delta x,$$

$$2\sigma_{x+\delta x} = 2\sigma + (u_{21}y + u_{22}z - p_{21}\varrho - p_{22}\sigma)\delta x,$$

where, of course, u_{12} and u_{21} may be equated to zero, since C_y and C_z are the two branches of the flecnode curve.

Now clearly, the coefficients of y, z, ϱ, σ in the expressions

$$\varrho_{x+\delta x} + \lambda y_{x+\delta x} \quad \text{and} \quad \sigma_{x+\delta x} + \mu z_{x+\delta x}$$

will be the coordinates of two arbitrary points P_1 and P_2, situated on $f'_{x+\delta x}$ and $f''_{x+\delta x}$ respectively. If (27) is the equation of the osculating complex, the plane which corresponds in it to P_1 must contain P_2 for arbitrary values of λ and μ. This consideration will enable us to determine the ratio $a:b$.

We find first, remembering that $u_{12} = u_{21} = 0$, for P_1 and P_2 the coordinates:

	x_1	x_2	x_3	x_4
P_1	$2\lambda + (u_{11} - \lambda p_{11})\delta x$	$-\lambda p_{12}\delta x$	$2 + (-p_{11} + \lambda)\delta x$	$-p_{12}\delta x$
P_2	$-\mu p_{21}\delta x$	$2\mu + (u_{22} - \mu p_{22})\delta x$	$-p_{21}\delta x$	$2 + (-p_{22} + \mu)\delta x$

But, if we denote by $u_1, \ldots, u_4$, the coordinates of the plane which corresponds to the point $x_1, \ldots, x_4$ in the linear complex (27), we find

$$u_1 = -a x_3, \quad u_2 = +b x_4, \quad u_3 = a x_1, \quad u_4 = -b x_2.$$

Substituting for $x_1, \ldots, x_4$ the coordinates of P_1, and writing down the condition that P_2 shall lie in the plane corresponding to P_1, we find that we must have

$$a(\mu - \lambda) p_{21} + b(\lambda - \mu) p_{12} = 0$$

for arbitrary values of λ and μ, i. e., $a : b = p_{12} : p_{21}$.

Therefore, *the equation of the osculating linear complex, in the system of coordinates here employed, is*

$$p_{12}\omega_{13} + p_{21}\omega_{12} = 0. \tag{28}$$

The point-plane correspondence, determined by this complex, is given by the equations

$$u_1 = p_{12} x_3, \quad u_2 = -p_{21} x_4, \quad u_3 = -p_{12} x_1, \quad u_4 = p_{21} x_2. \tag{29}$$

Let us consider a point on the generator g. There will correspond to it, in this complex, a plane, obviously containing the generator itself. But to every point of g there also corresponds another plane through g, viz., the plane tangent to the ruled surface at that point. Clearly, there will exist in general two points on g at which these two planes will coincide. We shall call them the *complex points* of g, and their locus on S, the *complex curve* of the surface. We proceed to determine the complex points of g.

The plane corresponding to any point of g, $(x_1, x_2, 0, 0)$, in the osculating linear complex, has the coordinates

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = -p_{12} x_1, \quad u_4 = p_{21} x_2. \tag{30}$$

The coordinates of the plane, tangent to S at the same point, are found most easily by computing the equation of the plane tangent to the osculating hyperboloid H at that point. They are $(0, 0, x_2, -x_1)$. This plane and (30) coincide if and only if $-p_{12} x_1 : p_{21} x_2 = x_2 : -x_1$, i. e., if

$$p_{12} x_1^2 - p_{21} x_2^2 = 0. \tag{31}$$

This shows that *the complex points and the flecnodes form a harmonic group on every generator of the surface.*

If in (7) we put $u_{11} + u_{22} = 0$, we find that the derivative cubic intersects g precisely in the complex points. Therefore: *if the surface S' of the congruence Γ is so chosen that it intersects the flecnode surface of S in an asymptotic curve, the surface of derivative cubics will intersect S along its complex curve. If S is contained in a linear complex, the complex curve is at the same time an asymptotic curve.*

This last statement follows from our previous results, but may be verified directly as follows. We notice in the first place that, under our assumptions, the factors of the expression $p_{12}z^2 - p_{21}y^2$ determine the complex points. Let us assume $p_{11} - p_{22} = 0$, which we may do without affecting the generality of our argument. Then the condition $\Delta = 0$ for a linear complex becomes $\frac{p_{12}}{p_{21}} = \text{const}$. If we now make the transformation

$$\bar{y} = \sqrt{p_{21}}\,y + \sqrt{p_{12}}\,z, \quad \bar{z} = \sqrt{p_{21}}\,y - \sqrt{p_{12}}\,z,$$

we find that, in the transformed system of differential equations, $\pi_{12} = \pi_{21} = 0$, the coefficients of this transformed system being denoted by Greek letters. But this proves that C_y and C_z are asymptotic lines on S. It is geometrically evident that the tangents of this asymptotic curve will be lines of the linear complex.

Lie apparently was the first to notice the existence of this special asymptotic curve on a ruled surface belonging to a linear complex. He proved, in 1871, that its determination requires no integration, and that all other asymptotic curves may be obtained by quadratures.[1]) These latter remarks we can also verify at once from our theory. *Picard* found the same theorems independently in 1877.[2]) These results on the determination of all of the asymptotic curves by quadratures if one of them is known, follow at once from the fact first noted by *Bonnet* that their equation is of the *Riccati* form, and had already been explicitly formulated and applied to special surfaces by *Clebsch.*[3]) It seems that *Voss*[4]) was the first to notice that this asymptotic curve and the flecnode curve divide the generators of the surface harmonically. *Cremona*[5]) however had already made this observation in the special case of a unicursal surface with two straight

1) *Lie,* Verhandl. d Ges. d. Wiss Christiania (1871), Mathematische Annalen, vol. 5 (1872).

2) *Picard, These,* Paris (1877) See also Darboux Bulletin (1877), p. 335, and Annales de l'École Normale (1877)

3) *Clebsch,* Crelle's Journal, vol. 68.

4) *Voss,* Mathematische Annalen, vol. 8 (1875).

5) *Cremona,* Annali di Matematiche (1867—68)

line directrices. *Halphen* and *Snyder*[1]) extended this theorem of *Cremona*'s to all surfaces with two straight line directrices. The general notion of the complex curve, its relation to the derivative surface and to the surface of derivative cubics seem to occur for the first time in a paper by the author.[2])

There always exists a pair of points harmonically conjugate to each of two given pairs. We see that the pair

$$p_{21}y^2 + p_{12}z^2 \tag{32}$$

is thus situated with respect to the flecnodes and the complex points. They are, therefore, the double points of an involution of which the flecnodes and the complex points are two pairs. We will call them the *involute points*, their locus the *involute curve*. Of course these three pairs cannot be real at the same time.

Consider the covariant of weight 7,

$$C_7 = \Theta_4 E - \Theta_4' C, \tag{33}$$

where

$$C = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz, \quad E = v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz.$$

We have seen in Chapter IV that C_7 is a covariant. Moreover it reduces to (32) under our special assumptions. Therefore, *the factors of the covariant $\Theta_4 E - \Theta_4' C$ give the expressions for the involute points in invariant form.* If

$$\Theta_4' u_{12} \quad \Theta_4 v_{12} = 0, \quad \Theta_4' u_{21} - \Theta_4 v_{21} = 0, \tag{34}$$

the combined locus of C_y and C_z constitutes the involute curve. If however

$$u_{11} - u_{22} = 0, \quad v_{11} - v_{22} = 0, \tag{35}$$

then C_y and C_z together constitute the complex curve.

We can also write down a covariant whose factors give the complex points. It is found by expressing the conditions that a quadratic in y and z shall represent points which divide the pair of the flecnodes, and the pair of involute points harmonically. We find in this way, that the *factors of*

$$\begin{aligned}[(u_{11} - u_{22})v_{12} - (v_{11} - v_{22})u_{12}]z^2 + [(u_{11} - u_{22})v_{21} \\ - (v_{11} - v_{22})u_{21}]y^2 + 2[u_{12}v_{21} - u_{21}v_{12}]yz\end{aligned} \tag{36}$$

represent the complex points.

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1) *Halphen*, Bull. Soc. Math. de France, vol. V (1877). *Snyder,* Bulletin of Am. Math. Soc., vol. V (1899).

2) *Wilczynski*, Transactions of the American Mathematical Society (1904), vol. 5, p. 243.

We can easily show that the derivative cubic cannot intersect g in its involute points unless S is a quadric. Moreover, if the derivative cubic intersects g in two points which are harmonic conjugates with respect to the complex point, S can only be a quadric. The cubic may, however, intersect g in two points harmonic conjugates with respect to the involute points, provided that $u_{11} + u_{22} = 0$, i. e., provided that S' intersects the flecnode surface of S in an asymptotic curve. It then passes through the complex points.

It is further clear, geometrically as well as analytically, that the two complex points can coincide only if S has a straight line directrix or if the flecnodes coincide. They become indeterminate if S has two straight line directrices. The involute points coincide if $\Theta_4 \neq 0$, and if S has a straight line directrix. They are indeterminate for $\Theta_4 = 0$.

To every point P' of g' there corresponds a plane in the osculating linear complex, as well as the plane tangent to S' at P'. When do these planes coincide?

Let the coordinates of P' be $(0, 0, \alpha_1, \alpha_2)$. The plane, corresponding to P' in the complex, has the coordinates $(\alpha_1 p_{12}, -\alpha_2 p_{21}, 0, 0)$, so that it contains g'. The coordinates of the plane tangent to S' at P' are, of course, the same as those of the plane tangent to H' at P', which may be obtained from (5), viz.: $(u_{11} u_{22}^2 \alpha_2, -u_{11}^2 u_{22} \alpha_1, 0, 0)$. These planes coincide if and only if

$$u_{11} p_{12} \alpha_1^2 = u_{22} p_{21} \alpha_2^2.$$

The corresponding points on g are again harmonic conjugates with respect to the flecnodes, i. e., those asymptotic tangents of S which join g to the points of g', the planes corresponding to which in the linear complex are the planes tangent to S', are harmonic conjugates with respect to the flecnode tangents.

The planes which correspond to these two points of g', in the null-system of the cubic, do not contain g'.

§ 4. Relation of the osculating linear complex to the linear complex of the derivative cubic.

The equations of the two complexes are

$$(37)\qquad \begin{aligned} \Omega_1 &= 4(C^2 - BD)\,\omega_{12} + AB\omega_{13} + AD\omega_{42} \\ &\qquad + A^2\omega_{34} + 2AC\omega_{23} - 2AC\omega_{14} = 0, \\ \Omega_2 &= p_{12}\,\omega_{13} + p_{21}\,\omega_{42} = 0. \end{aligned}$$

Their simultaneous invariant is

$$(38)\qquad -2u_{11}u_{22}(u_{11} - u_{22})^2 (u_{11} + u_{22})\, p_{12} p_{21}, \text{[1]}$$

which, leaving aside the cases when S' is developable or when S has one or more straight line directrices, vanishes if and only if $u_{11} + u_{22} = 0$.

1) Cf. chapter VII, § 3.

Therefore, *the osculating linear complex and the complex of the derivative cubic are in involution if the first derivative ruled surface cuts out asymptotic curves on the flecnode surface of S, and the cubic passes through the complex points of g.* Some of our previous theorems are consequences of this.

The two special complexes which are contained in the family

$$\lambda \Omega_1 + \mu \Omega_2 = 0,$$

where λ and μ are constants, are those for which

$$-3A^2BD\lambda^2 - 2u_{11}u_{22}(u_{11} - u_{22})^2(u_{11}+u_{22})p_{12}p_{21}\lambda\mu + p_{12}p_{21}\mu^2 = 0,$$

or, discarding again the case when S has a straight directrix,

$$-12u_{11}^3u_{22}^3(u_{11}-u_{22})^4\lambda^2 - 2u_{11}u_{22}(u_{11}-u_{22})^2(u_{11}+u_{22})\lambda\mu + \mu^2 = 0.$$

They coincide if

$$u_{11}^2u_{22}^2(u_{11}-u_{22})^4(u_{11}+u_{22})^2 + 12u_{11}^3u_{22}^3(u_{11}-u_{22})^4 = 0,$$

i. e., if S' is developable, if S is a quadric, or if

(39) $$(u_{11}+u_{22})^2 + 12u_{11}u_{22} = 0$$

We can always choose the independent variable so as to satisfy this condition. In fact, if we change the independent variable by putting $\xi = \xi(x)$, according to Chapter IV, equations (49), we shall have

$$(\bar{u}_{11}+\bar{u}_{22})^2 + 12\bar{u}_{11}\bar{u}_{22} = 0,$$

if ξ be taken as any solution of the equation

(40) $$64\mu^2 + 32(u_{11}+u_{22})\mu + (u_{11}+u_{22})^2 + 12u_{11}u_{22} = 0,$$

where

$$\mu = \{\xi, x\} = \eta' - \frac{1}{2}\eta^2, \quad \eta = \frac{\xi''}{\xi'}.$$

Therefore, there exist two families of ∞^1 non-developable ruled surfaces in the congruence Γ such that the linear congruence, common to the osculating linear complex of S and the linear complex of the derivative cubic, shall have coincident directrices. Any four surfaces of one family intersect all of the asymptotic tangents of S in a point row of constant anharmonic ratio. The two families never coincide unless $\Theta_4 = 0$, i. e., unless the flecnode curve intersects every generator in two coincident points. But in this case the congruence is not defined. If S has a straight line directrix this congruence is degenerate

The coordinates of the plane, which corresponds to a point $(x_1, x_2, 0, 0)$ of g in the null-system of the cubic, are

$$4(C^2 - BD)x_2, \quad -4(C^2-BD)x_1, \quad -ABx_1 - 2ACx_2, \quad 2ACx_1 + ADx_2.$$

This plane contains g if and only if $C^2 - BD = 0$, i. e., if the

derivative cubic is tangent to g. It will coincide with the plane tangent to S at this point, if further

$$-ABx_1 - 2ACx_2 = \omega x_2, \quad 2AC'x_1 + ADx_2 = -\omega x_1,$$

where ω is a proportionality factor, or

$$ABx_1 + (2AC' + \omega)x_2 = 0, \quad (2AC + \omega)x_1 + ADx_2 = 0,$$

whence follows $\omega = -AC'$ or $-3AC$. We have therefore

$$x_1 : x_2 = -C' : B = -D : C' \quad \text{or} \quad x_1 : x_2 = C' : B = D : C'.$$

These points are harmonic conjugates with respect to the flecnodes

Therefore, if the derivative cubic is tangent to g, there are two points of g whose tangent planes are the planes corresponding to them in the null-system of the cubic. These points and the flecnodes form a harmonic group on g. They never coincide with the complex points unless the ruled surface has a straight line directrix.

The planes, corresponding to a point of g in the null-system of the cubic and in the osculating complex, coincide if

$$\begin{aligned} &C'^2 - BD = 0, \\ &(AB - \omega p_{12})x_1 + 2AC'x_2 = 0, \\ &2ACx_1 + (AD - \omega p_{21})x_2 = 0, \end{aligned}$$

where ω is a root of the quadratic

$$(41) \quad \omega^2 + 2u_{11}u_{22}(u_{11} - u_{22})^2(u_{11} + u_{22})\omega - 12u_{11}^3u_{22}^3(u_{11} - u_{22})^4 = 0,$$

neglecting again the case when S has a straight line directrix. These two points of g coincide if $(u_{11} + u_{22})^2 + 12u_{11}u_{22} = 0$.

More generally, if we write down the conditions that the same plane shall correspond to a point (x_1, x_2, x_3, x_4) in the osculating linear complex and in the complex of the cubic, we shall obtain as the locus of these points two straight lines, the directrices of the congruence common to the two complexes. These conditions are as follows; $x_1 \dots x_4$ must satisfy the equations:

$$(42) \quad \begin{aligned} * \quad &+ 4(C'^2 - BD)x_2 + (AB - \omega p_{12})x_3 - 2AC'x_4 = 0, \\ -4(C'^2 - BD)x_1 &+ \quad * \quad + 2ACx_3 - (AD - \omega p_{21})x_4 = 0, \\ -(AB - \omega p_{12})x_1 &- 2AC'x_2 + \quad * \quad + A^2x_4 = 0, \\ 2AC'x_1 &+ (AD - \omega p_{21})x_2 - A^2x_3 + \quad * \quad = 0, \end{aligned}$$

the vanishing of whose skew-symmetric determinant gives for ω the quadratic equation (41), which may also be written

$$(41\text{a}) \quad (AB - \omega p_{12})(AD - \omega p_{21}) + 4A^2(C'^2 - BD) - 4A^2C^2 = 0.$$

Let ω_1 and ω_2 be the two roots of this equation. If we eliminate x_3 from the first two, x_4 from the last two equations of (42), if we make use of (41 a) and assume that neither A nor $C^2 - BD$ is zero, we shall find

$$\begin{aligned} &-(AB-\omega_k p_{12})x_1 - 2ACx_2 + A^2x_4 = 0, \\ &-4(C^2-BD)x_2 - (AB-\omega_k p_{12})x_3 + 2ACx_4 = 0 \end{aligned} \quad (k=1,2),$$

whence

$$\begin{aligned} &-2C(AB-\omega_k p_{12})x_1 - 4ABDx_2 \\ &\qquad + A(AB-\omega_k p_{12})x_3 = 0, \quad (k=1,2), \\ &-(AB-\omega_k p_{12})x_1 - 2ACx_2 + A^2x_4 = 0 \end{aligned} \tag{43}$$

the equations of the two directrices in simpler form than in (42).

A line joining the point $(x_1, x_2, 0, 0)$ of g to the point $(0, 0, x_1, x_2)$ of g' is a generator of the second kind on H. It is not difficult to see that it will intersect the directrix (43) if and only if

$$-(AB-\omega_k p_{12})^2x_1^2 + 4A^2BDx_2^2 = 0. \tag{44}$$

Hence, the two points, in which either of the directrices of the congruence common to the two complexes intersects the osculating hyperboloid, determine upon this hyperboloid two generators of the second set which are harmonic conjugates with respect to the flecnode tangents.

It also follows easily that the two pairs thus obtained, one corresponding to each directrix, coincide only if

$$(u_{11}+u_{22})^2 + 12u_{11}u_{22} = 0,$$

i. e., if the directrices themselves coincide. Further, if one of these pairs intersects g in the involute points, the same is true of the other pair, so that this can only happen if the directrices coincide. Finally, such a pair of generators of H can pass through the complex points only if S has a straight line directrix, or if S' is developable.

The line joining the points $(x_1, 0, x_3, 0)$ and $(0, x_1, 0, x_3)$ is a generator of the first set on H. The coordinates of an arbitrary point of this line are $(\lambda x_1, \mu x_1, \lambda x_3, \mu x_3)$ This line will, therefore, intersect one of the directrices of the congruence if x_1, x_3, λ, μ can be determined so as to satisfy the equations

$$\begin{aligned} &\lambda[-2C(AB-\omega_k p_{12})x_1 + A(AB-\omega_k p_{12})x_3] - 4\mu ABDx_1 = 0, \\ &-\lambda(AB-\omega_k p_{12})x_1 + \mu(-2ACx_1 + A^2x_3) = 0, \end{aligned}$$

which gives either $A=0$, $AB-\omega_k p_{12}=0$, or

$$4(C^2-BD)x_1^2 - 4ACx_1x_3 + A^2x_3^2 = 0 \tag{45}$$

The first two cases give either a surface S with a straight line directrix, or else a developable surface S'. Leaving these cases aside,

we notice that (45) does not contain ω_k so that if the line on H here considered intersects one of the directrices it intersects the other also. Combining this with our previous result, we see that the following theorem holds.

The four points in which the directrices of the congruence, common to the osculating linear complex and the linear complex of the derivative cubic, intersect the osculating hyperboloid can be grouped into two pairs, such that the line joining the members of each pair shall be a generator of the first set upon the hyperboloid. Upon this generator this pair of points, together with the intersections of the generator with the flecnode tangents, form a harmonic group.

The plane, corresponding to a point $(x_1, x_2, 0, 0)$ of g in the null-system of the cubic, intersects the flecnode tangents

$$f' \text{ in the point } [ABx_1 + 2ACx_2,\ 0,\ 4(C^2 - BD)x_2,\ 0],$$

$$f'' \text{ in the point } [0,\ 2ACx_1 + ADx_2,\ 0,\ 4(C^2 - BD)x_1].$$

The line joining these points is a generator of H, if either

$$A = 0, \quad \text{or} \quad C^2 - BD = 0, \quad \text{or} \quad Bx_1^2 - Dx_2^2 = 0.$$

Therefore, *there exist in general two points on g, harmonic conjugates with respect to the flecnodes, such that the planes, corresponding to them in the null-system of the derivative cubic, pass through a generator of the osculating hyperboloid.* If the cubic is tangent to g the null-plane of *any point* of g contains a generator of H, viz., g itself. If $A = 0$ likewise, *all points* of g satisfy the condition of the theorem. Their null-planes all pass through g'.

§ 5. Various theorems concerning the flecnode surface. The principal surface of the congruence.

Let us consider the planes which osculate the flecnode curve of S at P_y and P_z. We have, of course, $u_{12} = u_{21} = 0$. If x_1, x_2, x_3, x_4 are the coordinates of an arbitrary point of the plane osculating C_y at P_y, we have for the equation of this plane

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \end{vmatrix} = 0.$$

But

$$2y_k' = \rho_k - p_{11}y_k - p_{12}z_k, \quad 2z_k' = \sigma_k - p_{21}y_k - p_{22}z_k,$$

$$-y_k'' = p_{11}y_k' + p_{12}z_k' + q_{11}y_k + q_{12}z_k, \text{ etc.}$$

If we assume again $p_{11} = p_{22} = 0$, and substitute into the above equation, it becomes

$$\begin{vmatrix} x_1, & x_2, & \dots \\ y_1, & y_2, & \dots \\ \varrho_1 - p_{12} z_1, & \varrho_2 - p_{12} z_2, & \dots \\ p_{12}\sigma_1 + 2 q_{12} z_1, & p_{12}\sigma_2 + 2 q_{12} z_2, & \dots \end{vmatrix} = 0.$$

But if we introduce again the fundamental tetrahedron $P_y P_z P_\varrho P_\sigma$, this reduces to

$$p_{12} x_2 + p_{12}^2 x_3 - 2 q_{12} x_4 = 0. \tag{46a}$$

In the same way we find the equation of the plane osculating C'_z at P_z to be

$$p_{21} x_1 - 2 q_{21} x_3 + p_{21}^2 x_4 = 0. \tag{46b}$$

To these equations must be added the conditions $u_{12} = u_{21} = 0$, if C'_y and C'_z are the two branches of the flecnode curve.

Let us assume that this is the case. The osculating planes at P_y and P_z intersect along a straight line, whose intersections with the osculating hyperboloid may now be found. If $x_1 \dots x_4$ are the coordinates of one of these points of intersection, we find

$$\begin{aligned} p_{12} p_{21}^2 x_2 &= -2 q_{12} p_{21} x_1 - (p_{12}^2 p_{21}^2 - 4 q_{12} q_{21}) x_3, \\ p_{12} p_{21}^2 x_4 &= -p_{12} p_{21} x_1 + 2 q_{21} p_{12} x_3, \end{aligned} \tag{47}$$

where the ratio of $x_1 : x_3$ is determined by the quadratic

$$(p_{12}^2 p_{21}^2 - 4 q_{12} q_{21}) x_3^2 + 2 (p_{12} q_{21} + p_{21} q_{12}) x_1 x_3 - p_{12} p_{21} x_1^2 = 0. \tag{48}$$

Therefore, if $p_{12} q_{21} + p_{21} q_{12} = 0$, i. e., *if* $p_{12} p_{21} = const.$, *the two generators of the first kind on* H *which pass through these points are harmonic conjugates with respect to* g *and* g'. *If*

$$(p_{12} q_{21} - p_{21} q_{12})^2 + p_{12}^3 p_{21}^3 = 0, \tag{49}$$

the intersection of the two osculating planes is tangent to the hyperboloid. This latter property is obviously characteristic of a class of ruled surfaces, and can be expressed in invariant form.

We may find the invariant expression for this condition by making use of equations (102) of chapter IV. We there found that the coefficients of system (A) could be expressed as functions of the invariants, if the system is supposed to be reduced to the form characterized by the conditions $u_{12} = u_{21} = 0$ and $p_{11} = p_{22} = 0$. This is precisely the form in which we have supposed system (A) to be written in deducing (48) and (49). If we substitute the values of $p_{12}, p_{21}, q_{12}, q_{21}$ from the equations just mentioned into (49), we find

$$\Theta_4^3 \Theta_9^2 + 16 \Theta_{10}^3 = 0, \tag{49a}$$

while $p_{12} p_{21}$ becomes $\frac{\Theta_{10}}{\Theta_4^2}$. We have, therefore, the following results, which may also be verified *a posteriori*.

If the independent variable is chosen so that $\frac{\Theta_{10}}{\Theta_4^{\,2}} = const.$, *the generator g' of the first derived surface is the harmonic conjugate of g with respect to the two generators of the same kind on H which are determined by the points, in which the line of intersection of the two osculating planes of the flecnode curve, intersects H.*

If $\Theta_4^{\,3}\,\Theta_9^{\,2} + 16\,\Theta_{10}^{\,3} = 0$, *the ruled surface S has the following characteristic property. The planes, which osculate the flecnode curve at the two points of its intersection with any generator, intersect in a line which is tangent to the osculating hyperboloid.*

The plane osculating the flecnode curve at P_y intersects the flecnode tangent f'' which passes through P_z in the point $(0, 2q_{12}, 0, p_{12})$. Similarly, the plane osculating C_z at P_z intersects f' in the point $(2q_{21}, 0, p_{21}, 0)$. Therefore, the line joining these points is a generator of H if and only if $p_{12}q_{21} - p_{21}q_{12} = 0$, i. e., if S belongs to a linear complex. In other words: *the points in which the two planes, osculating the flecnode curve at its points of intersection with any generator, intersect the flecnode tangents are situated upon the same generator of the osculating hyperboloid if and only if the surface belongs to a linear complex.*

To each of the two planes (46a) and (46b) corresponds a point in that plane by means of the osculating linear complex. These points have the coordinates

$$(-p_{12}p_{21},\ -2q_{12},\ 0,\ -p_{12}) \quad \text{and} \quad (2q_{21},\ p_{12}p_{21},\ p_{21},\ 0).$$

We find that the line joining them intersects H in two points, which form a harmonic group with the first two, if S belongs to a linear complex. It is tangent to H if (49) is satisfied. *Therefore, if the two planes, osculating the flecnode curve at its two points of intersection with a generator, intersect in a line which is tangent to the osculating hyperboloid, the line joining the two points of these planes, which correspond to them in the osculating linear complex, is also tangent to the osculating hyperboloid, and conversely.*

We have seen that, under the assumptions

$$u_{12} = u_{21} = p_{11} = p_{22} = 0,$$

the equations of the sheet F' of the flecnode surface assume the form (15). Let us denote by $\mathfrak{U}_{ik}$ the quantities formed for this system according to the same law as are the quantities u_{ik} for the equations of S. Then we shall have, [cf. equations (4) chapter IX],

$$\mathfrak{U}_{12} = 0,\ \mathfrak{U}_{21} = -4(q_{11}' - q_{22}') + 8\,\frac{(q_{11} - q_{22})\,q_{12}}{p_{12}},\ \mathfrak{U}_{11} - \mathfrak{U}_{22} = 4(q_{11} - q_{22}).$$

The curve C_y is a branch of the flecnode curve on F' as well as on S. The other branch is the locus of the point

$$\mathfrak{U}_{21}\,y - (\mathfrak{U}_{11} - \mathfrak{U}_{22})\,\varrho.$$

Now if the transformation $\xi = \xi(x)$ is made, ϱ is converted into

$$\bar{\varrho} = \frac{1}{\xi'}(\varrho + \eta y), \quad \text{where} \quad \eta = \frac{\xi''}{\xi'}.$$

Therefore, if a transformation $\xi_1 = \xi_1(x)$ be made such that the derivative surface of S with respect to ξ_1 may cut out upon F' the second branch of its flecnode curve, ξ_1 must be so chosen that

(50a) $$\eta_1 = \frac{q_{11}' - q_{22}'}{q_{11} - q_{22}} - 2\frac{q_{12}}{p_{12}}.$$

Similarly the second branch of the flecnode curve on F'' will be obtained by putting

(50b) $$\eta_2 = \frac{q_{11}' - q_{22}'}{q_{11} - q_{22}} - 2\frac{q_{21}}{p_{21}}.$$

The two surfaces of Γ thus obtained coincide only if S belongs to a linear complex, i. e., *the second branches of the flecnode curves on the two sheets of the flecnode surface of S correspond to each other only if S belongs to a linear complex.*

We have seen that the plane osculating C_y at P_y intersects f'' in the point $2q_{12}z + p_{12}\sigma$. The corresponding point on f', i. e., the point obtained by finding the intersection of f' with the corresponding generator of H, is given by $2q_{12}y + p_{12}\varrho$. We find a surface of Γ which intersects f' and f'' in these points by making a transformation of the independent variables for which $\eta = 2\frac{q_{12}}{p_{12}}$, or if we denote this special value of η by $\bar{\eta}_1$,

$$\bar{\eta}_1 = 2\frac{q_{12}}{p_{12}}.$$

If we now denote by η the expression

$$\eta = \frac{1}{2}\frac{q_{11}' - q_{22}'}{q_{11} - q_{22}},$$

we find

$$\eta = \frac{1}{2}(\eta_1 + \bar{\eta}_1),$$

and similarly

$$\eta = \frac{1}{2}(\eta_2 + \bar{\eta}_2).$$

This gives us an important result. For, we have occasionally made use of a normal form for our system of differential equations, in which $\Theta_4 = \text{const}$. But in order to have $\Theta_4 = \text{const.}$, we must make precisely the transformation determined by η. On account of its importance, we shall call the surface of Γ which is thus obtained, the *principal surface* of the congruence, and the curves in which it intersects the two sheets of the flecnode surface of S their *principal*

curves. We see then that the principal surface may be constructed as follows.

We consider the flecnodes P_y and P_z of g, the planes p_y and p_z osculating the flecnode curve at these points, and the points P' and P'' upon the flecnode tangents f' and f'' whose loci are the second branches of the flecnode curves on the two sheets F' and F'' of the flecnode surface. The plane p_y intersects f'' in a certain point to which corresponds a point on f' such that the line joining them is a generator of the osculating hyperboloid. This latter point together with P' constitute a pair, such that the harmonic conjugate of P_y with respect to it is the point in which the principal surface intersects f'. The intersection with f'' is found in the same way.

We might also say, that in this way there is determined, upon the generators of H, an involution whose double elements are g and the generator of the principal surface.

By combining a number of our previous results with the notion of the principal surface, we obtain a number of theorems, which may be easily verified. They provide interpretations for the vanishing of certain invariants, and therefore furnish characteristic properties of certain families of ruled surfaces.

To prove these theorems it is sufficient to express the conditions in terms of the coefficients p_{ik}, q_{ik} and then introduce the invariants by means of the equations (102) of chapter IV.

If $\Theta_{15} = 0$, $\Theta_4 \neq 0$, the principal surface is the harmonic conjugate of S with respect to the two ruled surfaces of the flecnode congruence which cut out the second branches of the flecnode curves on F' and F''.

If $\Theta_{4\cdot 1} = 0$, or $\Theta_6\Theta_4 - 9\Theta_{10} = 0$, the principal surface intersects F' and F'' along asymptotic curves.

If $\Theta_{4\cdot 1}^2 - 64\,\Theta_4{}^5 = 0$, the principal surface is developable.

If $\Theta_{15}{}^2 - \Theta_9{}^2\,\Theta_4{}^3 = 0$ the principal surface intersects one of the sheets of the flecnode surface along the second branch of its flecnode curve. It thus intersects both sheets if $\Theta_{15} = \Theta_9 = 0$.

Our construction of the principal surface becomes indeterminate if S has two straight line directrices. If we assume that C_y and C_z are two asymptotic curves upon S, we may put, in this case,

$$p_{ik} = 0, \quad q_{12} = aq, \quad q_{21} = bq, \quad q_{11} - q_{22} = cq,$$

so that Θ_4 is a constant, if and only if q is a constant

The planes tangent to S' at P_ϱ and P_σ intersect P_yP_z in the two points

$$u = q_{11}y + aqz, \quad v = bqy + q_{22}z$$

respectively. The point $P_{u'}$ determined by

$$u' = \frac{du}{dx} = q_{11}{}'y + aq'z + \frac{1}{2}q_{11}\varrho + \frac{1}{2}aq\sigma$$

is a point upon the tangent of C_u at P_u. If the independent variable

is transformed by putting $\xi = \xi(x)$, the curve C_u is converted into another curve, and the point $P_{u'}$ is transformed into another point of its tangent. But the above equation shows that $P_{u'}$ lies in the plane $P_u P_\varrho P_\sigma$ if and only if $q' = 0$, i. e. if and only if the independent variable has been chosen so as to make S' the principal surface of the congruence $P_{u'}$ will then be in the plane $P_z P_\varrho P_\sigma$. This property may serve, provisionally, to characterize the principal surface in this case; another simpler interpretation will be given later.[1])

§ 6. The covariant C_3 for $\Theta_4 \neq 0$.

We have interpreted geometrically all of the fundamental covariants, except C_3. With the notions, now at our disposal, we may also find the significance of this covariant It is equal to

$$C_3 = E + 2N, \tag{51}$$

[cf. Chapter IV (116)], where E and N are defined by the equations (107) and (114) of Chapter IV. We may write

$$E = \begin{vmatrix} \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, & y \\ v_{21}y - \frac{1}{2}(v_{11} - v_{22})z, & z \end{vmatrix}, \tag{52}$$

$$N = \begin{vmatrix} (u_{11} - u_{22})\varrho + 2u_{12}\sigma, & y \\ 2u_{21}\varrho - (u_{11} - u_{22})\sigma, & z \end{vmatrix},$$

so that

$$C_3 = \alpha z - \beta y, \tag{53}$$

where

$$\alpha = 2(u_{11} - u_{22})\varrho + 4u_{12}\sigma + \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, \tag{54}$$

$$\beta = 4u_{21}\varrho - 2(u_{11} - u_{22})\sigma + v_{21}y - \frac{1}{2}(v_{11} - v_{22})z.$$

The covariant C_3, therefore, determines a ruled surface Σ, whose generator is obtained by joining the points P_α and P_β determined by (54). Our interpretation of the covariant C_3 will consist in giving a construction for this surface.

The surface Σ is *not* like the derivative surface S' of S dependent upon the choice of the independent variable. In fact, it may be easily verified that the transformation $\xi = \xi(x)$ converts α and β into $\bar{\alpha}$ and $\bar{\beta}$, where

$$\bar{\alpha} = \frac{1}{(\xi')^3}\alpha, \quad \bar{\beta} = \frac{1}{(\xi')^3}\beta,$$

so that the points P_α and P_β are left invariant. A transformation of the dependent variables

1) Chapter XIV, § 5

$$\bar{y} = ly + mz, \quad \bar{z} = ny + rz$$

transforms α and β cogrediently into

$$\bar{\alpha} = l\alpha + m\beta, \quad \bar{\beta} = n\alpha + r\beta,$$

so that there is a one-to-one correspondence between the points of the generators of S and Σ.

We may, therefore, in order to determine the surface Σ choose the curves C_y and C_z of S, and the independent variable in any way that may be convenient. We shall assume $\Theta_4 \neq 0$, so that the flecnode curve of S intersects the generators in distinct points, and identify C_y and C_z with the two branches of the flecnode curve, so that

$$u_{12} = u_{21} = 0$$

We shall further assume that the independent variable is chosen in such a way as to make $\Theta_4 = 1$, or more specifically (since $u_{12} = u_{21} = 0$), so that

$$u_{11} - u_{22} = 1.$$

The derivative surface S' of S with respect to x is then the principal surface of its flecnode congruence We find, with these assumptions,

$$\alpha = 2\varrho - p_{12}z, \quad \beta = -2\sigma + p_{21}y. \tag{55}$$

We have, on the other hand,

$$\varrho = 2y' + p_{11}y + p_{12}z, \quad \sigma = 2z' + p_{21}y + p_{22}z,$$

whence

$$\gamma = 2y' + p_{11}y = \varrho - p_{12}z, \quad \delta = 2z' + p_{22}z = \sigma - p_{21}y. \tag{56}$$

The point P_γ is obviously the intersection of the tangent to the flecnode curve at P_y with the line $P_z P_\varrho$, while P_δ is the intersection of the tangent to the flecnode curve at P_z with the line $P_y P_\sigma$.

But we have from (55) and (56)

$$\gamma = \frac{1}{2}(\alpha - p_{12}z), \quad \varrho = \frac{1}{2}(\alpha + p_{12}z),$$

$$\delta = -\frac{1}{2}(\beta + p_{21}y), \quad \sigma = -\frac{1}{2}(\beta - p_{21}y),$$

so that the points P_γ and P_ϱ are harmonic conjugates with respect to P_α and P_z on the line $P_z P_\varrho$, while P_δ and P_σ are harmonic conjugates with respect to P_β and P_y.

The generator of the surface Σ is now completely determined by the following construction.

Let P_y and P_z be the two flecnodes, supposed distinct, on a given generator of the ruled surface S, and let P_ϱ and P_σ be the points corresponding to P_y and P_z respectively upon the principal surface of

the flecnode congruence of S. At P_y, as well as at P_z, three important lines intersect, viz.: the generator, the flecnode tangent, and the tangent to the flecnode curve. All of these are in the plane tangent to the surface S at their point of intersection. In each of these plane pencils we construct a fourth line, the harmonic conjugate of the generator with respect to the other two. Each of these lines meets the line joining the point of the principal surface, which corresponds to the flecnode considered, to the other flecnode. The line, which joins the two points of intersection, P_α and P_β, obtained in this way, is the generator of Σ which corresponds to the given generator of S.

Examples.

Ex 1. If S has two straight line directrices, Σ coincides with the principal surface of Γ.

Ex. 2. If a ruled surface belongs to a linear congruence with distinct directrices, every asymptotic curve intersects each generator in two points which divide the flecnodes harmonically *(Cremona, Halphen, Snyder.)*

Ex. 3.* Find the conditions that an asymptotic curve, flecnode curve, involute curve, complex curve of S corresponds to one or the other of these curves on S'. Investigate these correspondences in detail

Ex 4. The covariant C_3 can vanish identically, only if S is a quadric

Ex 5.* Set up the differential equations of the surface Σ, and find the conditions that it may belong to a linear complex, a linear congruence, or be a quadric, etc... Express these conditions as invariant conditions for the surface S

CHAPTER XI.

RULED SURFACES WHOSE FLECNODE CURVE INTERSECTS EVERY GENERATOR IN TWO COINCIDENT POINTS

The formulae and the theorems developed in the preceding chapter are not directly applicable to the case when $\Theta_4 = 0$, i. e., when the flecnode curve intersects every generator in two coincident points. The general notions, employed there, may however be applied to this special case as well, and give rise to a number of interesting and important considerations.

§ 1. The covariant C_8.

The interpretation of the covariant C_8 given in the last chapter is complete for the case $\Theta_4 \neq 0$. It breaks down absolutely for $\Theta_4 = 0$.

This covariant is

$$C_8 = \alpha z - \beta y, \tag{1}$$

where

$$\begin{aligned} \alpha &= 2(u_{11} - u_{22})\varrho + 4u_{12}\sigma + \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, \\ \beta &= 4u_{21}\varrho - 2(u_{11} - u_{22})\sigma + v_{21}y - \frac{1}{2}(v_{11} - v_{22})z. \end{aligned} \tag{2}$$

We have

$$\Theta_4 = (u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0.$$

Let us assume that the curve C_y is the flecnode curve, so that $u_{12} = 0$. We shall then have also $u_{11} - u_{22} = 0$. Further we may assume $p_{11} = p_{22} = 0$.

We have therefore

$$u_{12} = u_{11} - u_{22} = 0, \quad p_{11} = p_{22} = 0,$$

whence

$$\alpha = p_{12}u_{21}y, \quad \beta = 4u_{21}\varrho + v_{21}y - p_{12}u_{21}z. \tag{3}$$

If a transformation of the independent variable be made by putting $\xi = \xi(x)$, we find that for the new system of differential equations

$$\bar{v}_{21} = \frac{1}{(\xi')^3}(v_{21} - 4u_{21}\eta), \quad \eta = \frac{\xi''}{\xi'}.\text{[1]}$$

Therefore, if $u_{21} \neq 0$, i. e., if S is not a quadric, we can always choose η in just one way so as to take $\bar{v}_{21} = 0$ We obtain, therefore a perfectly definite surface of the congruence Γ, which we will call its *principal* surface, and which we shall characterize geometrically farther on

Let us assume that the variable x has already been chosen in such a way that S', the derivative of S with respect to x, coincides with the principal surface of Γ. Then $v_{21} = 0$, and (3) becomes

$$\alpha = p_{12}u_{21}y, \quad \beta = 4u_{21}\varrho - p_{12}u_{21}z = u_{21}\bar{\beta},$$

where

$$\bar{\beta} = 4\varrho - p_{12}z.$$

We have further in general

$$\varrho = 2y' + p_{11}y + p_{12}z,$$

whence

$$2y' = \varrho - p_{12}z.$$

1) Chapter IV, equ. (53).

The point, whose coordinates are $y_1', \ldots, y_4'$, is, therefore, obviously the intersection of the tangent to the flecnode curve with the line $P_z P_\varrho$. The point P_β whose coordinates are given by $\beta_1 \ldots \beta_4$ is also on the line $P_z P_\varrho$, and the cross ratio of the four points $P_\varrho P_\beta P_z P_{y'}$ is $\frac{1}{4}$. The point P_α obviously coincides with P_y.

The ruled surface which the covariant C_3 adjoins to S may therefore be defined as follows. In the plane tangent to S at its flecnode P_y, construct a line passing through P_y such that it, together with the generator, the flecnode tangent and the tangent of the flecnode curve shall constitute a plane pencil whose anharmonic ratio is $\frac{1}{4}$. The locus of these lines is the required ruled surface. Moreover the points of any generator of this surface are, by means of the covariant C_3, put into a one-to-one correspondence with those of g. The lines joining corresponding points pass through P_ϱ, that point of the principal surface of the congruence Γ which corresponds to P_y

It only remains to give a characteristic geometric property of the principal surface of Γ. For this purpose let us assume, in addition to our previous hypotheses, that C_z is an asymptotic curve on S, i e., let $p_{21} = 0$. Then.

$$
(4) \qquad
\begin{aligned}
2\sigma' &= u_{21} y + u_{11} z, \\
2\sigma'' &= u_{21}' y + \left(u_{11}' - \frac{1}{2} u_{21} p_{12}\right) z + \frac{1}{2}\left(u_{21} \varrho + u_{11} \sigma\right),
\end{aligned}
$$

and

$$v_{21} = 2u_{21}'$$

The first equation shows that the tangent to the curve C_σ at P_σ intersects the generator g of S Denote this point of intersection by $P_{\sigma'}$. The locus of the point $P_{\sigma'}$ is therefore a curve on S, $C_{\sigma'}$ Its tangent at $P_{\sigma'}$ is obtained by joining $P_{\sigma'}$ to the point $P_{\sigma''}$ defined by the second equation (4). But this equation shows that $P_{\sigma''}$ is in the plane $P_z P_\varrho P_\sigma$ if and only if $v_{21} = 0$ Put

$$\tau = u_{21} \varrho + u_{11} \sigma;$$

then P_τ is a point on $P_\varrho P_\sigma$ such that the line joining it to $P_{\sigma'}$ is a generator of the hyperboloid H osculating S along g. We see that the tangent to $C_{\sigma'}$ always intersects $P_z P_\tau$, and that $P_{\sigma''}$ coincides with this point of intersection if and only if $v_{21} = 0$.

The principal surface of the congruence Γ is therefore defined by the following statement, which is merely provisional, however, as we shall find a simpler interpretation later.[1])

We draw upon the ruled surface S any asymptotic line C_z and, upon any surface S' of the congruence Γ, the curve C_σ which cor-

1) Chapter XIV, § 5.

responds to it, P_z and P_σ being corresponding points of the two curves. The tangent to C_σ at P_σ always intersects the generator g of S, which passes through P_z, in a certain point $P_{\sigma'}$, whose locus gives a curve $C_{\sigma'}$ upon S. Upon S' a point P_τ is constructed such that the line $P_\tau P_{\sigma'}$ shall be a generator of the hyperboloid osculating S along g. The tangent to $C_{\sigma'}$ at $P_{\sigma'}$ intersects the line $P_z P_\tau$. If the independent variable has been chosen in such a way that S' is the principal surface of the congruence, the point $P_{\sigma''}$ will coincide with this point of intersection.

§ 2. The derivative conic.

In the general case, where $\Theta_4 \neq 0$, the hyperboloid H osculating S along g and the hyperboloid H' osculating S' along g' intersect along g' and a space cubic, which we have called the derivative cubic. Moreover, this cubic does not degenerate unless either S has a straight line directrix or S' is developable.

In the present case, however, H and H' have besides g' the straight line f in common, i. e., the flecnode tangent passing through P_y. The rest of their intersection is, therefore, a conic which we shall call the *derivative conic*.

We proceed to prove these statements and to derive the equations of the derivative conic. Taking as tetrahedron of reference the tetrahedron $P_y P_z P_\varrho P_\sigma$, the equation of H is

$$x_1 x_4 - x_2 x_3 = 0,$$

and that of H'

$$(u_{22} x_1 - u_{21} x_2)\,[J x_4 - (\lambda_{12} u_{22} - \lambda_{22} u_{12}) x_1 + (\lambda_{12} u_{21} - \lambda_{22} u_{11}) x_2]$$
$$+ (u_{12} x_1 - u_{11} x_2)\,[J x_3 - (\lambda_{11} u_{22} - \lambda_{21} u_{12}) x_1 + (\lambda_{11} u_{21} - \lambda_{21} u_{11}) x_2] = 0,$$

as we have shown in Chapter X, equations (1) and (3).

In our case we may put

$$u_{12} = u_{11} - u_{22} = 0, \quad J = u_{11}^2,$$

whence

(5) $\quad v_{11} - v_{22} = 2 p_{12} u_{21}, \quad v_{12} = 0, \quad v_{21} = 2 u_{21}' - (p_{11} - p_{22}) u_{21}$

and

(6)
$$2J\lambda_{11} = - u_{11} v_{11}, \qquad 2J\lambda_{12} = 0,$$
$$2J\lambda_{21} = u_{21} v_{22} - u_{11} v_{21}, \quad 2J\lambda_{22} = - u_{11} v_{22}.$$

We find, therefore, for H' the equation

$$(u_{11} x_1 - u_{21} x_2)\,[J x_4 - \lambda_{22} u_{11} x_2]$$
$$- u_{11} x_2\,[J x_3 - \lambda_{11} u_{11} x_1 + (\lambda_{11} u_{21} - \lambda_{21} u_{11}) x_2] = 0,$$

or

$$(7)\qquad J u_{11}(x_1 x_4 - x_2 x_3) + u_{11}^2(\lambda_{11} - \lambda_{22}) x_1 x_2 - J u_{21} x_2 x_4 \\ + [u_{11}^2 \lambda_{21} - u_{11} u_{21}(\lambda_{11} - \lambda_{22})] x_2^2 = 0,$$

while the equation of H is

$$x_1 x_4 - x_2 x_3 = 0.$$

Both equations are satisfied by $x_1 = x_2 = 0$, as well as by $x_2 = x_4 = 0$, which proves that g' and f are lines upon both of these hyperboloids. They must therefore have also a conic in common, whose plane must, according to (7), have the equation

$$u_{11}(\lambda_{11} - \lambda_{22}) x_1 - u_{11} u_{21} x_4 + [u_{11}\lambda_{21} - u_{21}(\lambda_{11} - \lambda_{22})] x_2 = 0.$$

If we put for abbreviation

$$(8)\quad A = u_{11}(\lambda_{11} - \lambda_{22}), \quad B = -u_{11}\lambda_{21} + u_{21}(\lambda_{11} - \lambda_{22}), \quad C = u_{11} u_{21},$$

whence

$$(9)\qquad u_{11} A + p_{12} C = 0,$$

we have, therefore, as the *equations of the derivative conic*

$$(10)\qquad A x_1 - B x_2 - C x_4 = 0, \quad x_1 x_4 - x_2 x_3 = 0.$$

We may also express the coordinates of any point on the conic in terms of a parameter t. Any point on the hyperboloid H can be represented in the form

$$x_1 = ut, \quad x_2 = t, \quad x_3 = u, \quad x_4 = 1$$

This point is, moreover, a point of the conic if the condition

$$Aut - Bt - C = 0$$

is satisfied, whence

$$u = \frac{C + Bt}{At}.$$

If we substitute into the above equations for $x_1 \ldots x_4$ and multiply by At, we find

$$(11)\quad x_1 = t(C + Bt), \quad x_2 = At^2, \quad x_3 = A + Bt, \quad x_4 = At$$

as the parametric equations of the conic, or in homogeneous form

$$(12)\qquad x_1 = t_1\psi, \quad x_2 = At_1^2, \quad x_3 = t_2\psi, \quad x_4 = At_1 t_2,$$

where

$$(13)\qquad \psi = Bt_1 + Ct_2.$$

The conic, of course, always passes through P_ϱ. The first question which we naturally ask is this: when does the conic degenerate into a pair of lines? Clearly this can only happen if the plane

$$Ax_1 - Bx_2 - Cx_4 = 0$$

intersects the hyperboloid H in a pair of lines, i. e., if it is tangent to H. Moreover since this plane contains P_ϱ, it must in that case contain at least one of the two generators of H which pass through

P_ϱ. If it contains that one which passes also through P_σ we must have $C = 0$, i. e, since $u_{21} \neq 0$ (S not being an quadric), $u_{11} = 0$. S' must therefore be a developable. If, however, this plane contains the other generator through P_ϱ, namely that one which passes through P_y, we must have $A = 0$, which gives either $u_{11} = 0$ as before or else $p_{12} = 0$, in which case the flecnode curve C_y would be a straight line

Therefore, the derivative conics degenerate if and only if the surface S has a straight line directrix, or else if the derivative of S with respect to x is one of the developable surfaces of the congruence Γ.

By an investigation similar to that in Chapter X, we obtain the further result:

Two consecutive derivative conics never intersect unless they degenerate.

§ 3 The developable surface generated by the plane of the derivative conic.

As x changes, the plane of the conic C_x envelops a developable surface, the equations of whose generator we shall now proceed to determine.

Let us form $\partial\Phi/\partial x$ under the assumption that $t_1 : t_2$ is independent of x. Then

$$\Phi + \frac{\partial \Phi}{\partial x}\,\delta x$$

will represent any point on the derivative conic $C_{x+\delta x}$ belonging to the argument $x + \delta x$, where δx is an infinitesimal. The plane of this conic will be determined by any three points upon it. We have (again assuming $p_{21} = 0$),

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x} = y\Big[t_1(B't_1 + C't_2) &+ \tfrac{1}{2}(Bt_1 + Ct_2)u_{11}t_2 + \tfrac{1}{2}At_1t_2u_{21}\Big] \\
&+ z\Big[A't_1^2 - \tfrac{1}{2}(Bt_1 + Ct_2)p_{12}t_1 + \tfrac{1}{2}At_1t_2u_{11}\Big] \\
(14)\qquad &+ \varrho\Big[t_2(B't_1 + C't_2) + \tfrac{1}{2}(Bt_1 + Ct_2)t_1\Big] \\
&+ \sigma\Big[A't_1t_2 - \tfrac{1}{2}(Bt_1 + Ct_2)p_{12}t_2 + \tfrac{1}{2}At_1^2\Big]
\end{aligned}
$$

We can obtain three points of the conic $C_{x+\delta x}$ by putting $t_1 = 0$, $t_2 = 1$; $t_1 = 1$, $t_2 = 0$; $t_1 = +C$, $t_2 = -B$, which last set of values corresponds to $\psi = 0$. Therefore, the equation of the plane of $C_{x+\delta x}$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ Cu_{11}\delta x & 0 & 2C + 2C'\delta x & -Cp_{12}\delta x \\ 2B + 2B'\delta x & 2A + (2A' - Bp_{12})\delta x & B\delta x & A\delta x \\ \begin{matrix}[2C(B'C - BC') \\ - ABCu_{21}]\delta x\end{matrix} & \begin{matrix}2AC^2 + (2A'C^2 \\ - ABCu_{11})\delta x\end{matrix} & \begin{matrix}-2B(B'C \\ - BC')\delta x\end{matrix} & \begin{matrix}-2ABC + (AC'^2 \\ - 2A'BC)\delta x\end{matrix} \end{vmatrix} = 0$$

If this determinant be developed, retaining of course only the terms of the first order in δx, we find that the planes of C_x and of $C_{x+\delta x}$ intersect along the line

$$(15) \qquad \begin{aligned} Ax_1 - Bx_2 - Cx_4 &= 0, \\ ADx_1 - Ex_2 - CFx_4 &= 0, \end{aligned}$$

where

$$(16) \qquad \begin{aligned} D &= 4A'C + 2AC' - BCp_{12}, \\ E &= 2BCA' + 2ACB' + 2ABC' - AC'^2, \\ F &= 4AC'' + 2A'C' - ABu_{11} + A^2u_{21} = 4AC'' + 2A'C' + Au_{11}^2\lambda_{21}. \end{aligned}$$

Equations (15) *are, therefore, the equations of the generator of the developable surface*

This proof would not be valid if either C or B were zero. For, then the third of the three points of the conic $C_{x+\delta x}$, which we have used to determine its plane, would coincide with one of the other two. If $C = 0$ the conic degenerates. Since, however, we might, in the case $B = 0$, choose three other points of $C_{x+\delta x}$, as we might also do in the general case, the result will obviously be obtained from the general case by substituting $B = 0$.

We see from (15) that the generator of the developable surface passes through P_ϱ and that it intersects the plane $x_3 = 0$, in the point P_χ or

$$[C(E - BF),\ AC(D - F),\ 0,\ A(E - BD)],$$

which we may therefore represent by the expression

$$(17) \qquad \chi = C(E - BF)y + AC(D - F)z + A(E - BD)\sigma.$$

Since the surface generated by this line joining P_ϱ to P_χ is developable, it must be possible to find four functions $\alpha, \beta, \gamma, \delta$ of x, such that

$$(18) \qquad \alpha\varrho + \beta\chi + \gamma\varrho' + \delta\chi' = 0.$$

Now we have

$$\varrho' = \frac{1}{2}(u_{11}y - p_{12}\sigma),$$

and we find

$$(19) \qquad \chi' = Gy + Hz + M\varrho + N\sigma,$$

where

$$
\begin{aligned}
G &= C(E' - BF' - B'F) + C'(E - BF) + \tfrac{1}{2} u_{21} A(E - BD),\\
H &= (AC' + A'C)(D - F) + AC(D' - F') - \tfrac{1}{2} p_{12} C(E - BF)\\
&\qquad + \tfrac{1}{2} u_{11} A(E - BD),\\
M &= \tfrac{1}{2} C(E - BF),\\
N &= A'(E - BD) + A(E' - BD' - B'D) + \tfrac{1}{2} AC(D - F).
\end{aligned}
\tag{20}
$$

If we substitute these values of χ' and ϱ' and also the expression (17) for χ into (18), we find that $\alpha, \beta, \gamma, \delta$ must satisfy the equations

$$
\begin{aligned}
C(E - BF)\beta + \tfrac{1}{2} u_{11}\gamma + G\delta &= 0,\\
AC(D - F)\beta \qquad\qquad + H\delta &= 0,\\
A(E - BD)\beta - \tfrac{1}{2} p_{12}\gamma + N\delta &= 0,\\
\alpha + M\delta &= 0.
\end{aligned}
\tag{21}
$$

Therefore, the determinant of the first three equations, which expanded becomes

$$
\tfrac{1}{2} p_{12} C(D - F)[NC - GA + HB],
\tag{22}
$$

must vanish identically; i. e., since the other factors do not vanish identically, we must have

$$
NC - GA + HB = 0.
\tag{23}
$$

We may also verify (23) directly. For we find from (20),

$$
\begin{aligned}
NC - GA + HB = {} & F\left(ACB' - \tfrac{1}{2} AC^2 - BA'C + \tfrac{1}{2} B^2 C p_{12}\right)\\
& - D\left(ACB' - \tfrac{1}{2} AC^2 - BAC' - \tfrac{1}{2} A^2 B u_{21} + \tfrac{1}{2} AB^2 u_{11}\right)\\
& + E\left(A'C - AC' - \tfrac{1}{2} A^2 u_{21} - \tfrac{1}{2} CBp_{12} + \tfrac{1}{2} ABu_{11}\right)\\
& - \tfrac{1}{2} [F(E - BD) - D(E - BF) + E(D - F)] = 0.
\end{aligned}
$$

We can now determine the edge of regression of the developable surface. If $\gamma\varrho + \delta\chi$ is a point on this curve, its tangent constructed at that point must coincide with the generator of the developable, i. e.,

$$
\gamma'\varrho + \delta'\chi + \gamma\varrho' + \delta\chi' = \lambda\chi + \mu\varrho
\tag{24}
$$

or

$$
(\gamma' - \mu)\varrho + (\delta' - \lambda)\chi + \gamma\varrho' + \delta\chi' = 0,
$$

which is identical with (18) if we put there

$$
\alpha = \gamma' - \mu, \quad \beta = \delta' - \lambda.
$$

But, on account of (23), we can determine $\alpha, \beta, \gamma, \delta$ so as to satisfy (18); we can, therefore, determine $\lambda, \mu, \gamma, \delta$ so as to satisfy (24). Moreover we find

$$(25) \qquad \gamma : \delta = -2GA(D-F) + 2H(E-BF) : u_{11}A(D-F).$$

Therefore, the edge of regression of the developable is given by the expression

$$(26) \qquad \begin{aligned} z = [-2GA(D-F) &+ 2H(E-BF)]\varrho \\ &+ u_{11}A(D-F)[C(E-BF)y + AC(D-F)z \\ &+ A(E-BD)\sigma]. \end{aligned}$$

We see from (15) that the generator of the developable surface coincides with one of the generators of H which passes through P_ϱ, only if either A or C vanishes, i e., either if S' is developable or if S has a straight line directrix, in which cases the derivative conic degenerates. *The generator of the developable is tangent to H at P_ϱ, neglecting the cases just mentioned, only if $D - F = 0$.* As (26) *shows, the cuspidal edge of the developable then coincides with C_ϱ.* If the expressions for A, B, C be substituted into the condition $D - F = 0$, or

$$2(A'C - AC') - BCp_{12} + ABu_{11} - A^2u_{21} = 0,$$

it becomes

$$4\frac{u'_{11}}{u_{11}} - 2\frac{p'_{12}}{p_{12}} - 2\frac{u'_{21}}{u_{21}} = 0,$$

which gives on integration

$$\frac{u_{11}^2}{p_{12}u_{21}} = \text{const}.$$

If $E - BF = 0$, P_y lies in the plane $P_\varrho P_z P_\sigma$, and if $E - BD = 0$ in the plane $P_y P_z P_\varrho$.

It will clearly be possible to characterize special classes of ruled surfaces (for $\Theta_4 = 0$) by special properties of the developable surfaces here considered.

The relation of S to its flecnode surface F is especially close in this case ($\Theta_4 = 0$). In fact S is also the flecnode surface of F. Moreover, the same hyperboloid H which osculates S along g, also osculates F along the corresponding generator f of F. The congruence Γ', which belongs to the surface F in the same way as Γ does to S, is therefore made up of the generators of the second set on the osculating hyperboloids of S, those of the first set constituting the lines of the congruence Γ. All of these remarks follow easily from the equations of the flecnode surface which, under the assumptions $p_{11} = p_{22} = p_{21} = 0$, assume the form

$$(28) \qquad \begin{aligned} &y'' - 2\frac{q_{12}}{p_{12}}y' - \varrho' - q_{11}y + \frac{q_{12}}{p_{12}}\varrho = 0, \\ &\varrho'' + 4q_{11}y' - 2\frac{q_{12}}{p_{12}}\varrho' + \left[2q_{11}' - p_{12}q_{21} - 4q_{11}\frac{q_{12}}{p_{12}}\right]y - q_{11}\varrho = 0. \end{aligned}$$

Moreover, since S' is a developable surface of the congruence Γ if $q_{11} = 0$, and since (28) shows that C_ϱ is then an asymptotic curve on F, we see that the developable surfaces of the congruence intersect its focal surface F along asymptotic lines, as it should according to the general theory of congruences.

Examples.

Ex. 1. If S belongs to a linear congruence, with coincident directrices, the results of § 1 are modified. Discuss this case.

Ex. 2. Find and discuss the conditions that the developable of § 3 shall be a cone.

Ex. 3.* Find and discuss the conditions that the edge of regression of the developable of § 3 may be a curve belonging to a linear complex; a space cubic.

CHAPTER XII.

GENERAL THEORY OF CURVES ON RULED SURFACES.

§ 1. Relation between the differential equations of the surface and of the curves situated upon it.

Let a ruled surface be given by means of the system of differential equations

$$(1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

so that the curves C_y and C_z will be two curves upon it, the lines joining corresponding points of these two curves being generators of the surface.

We shall eliminate once z and once y, so as to obtain the linear differential equations of the fourth order which each of these functions must satisfy.

We have from (1), by differentiation,

$$(2)\qquad \begin{aligned} y^{(3)} &= r_{11}y' + r_{12}z' + s_{11}y + s_{12}z, \\ z^{(3)} &= r_{21}y' + r_{22}z' + s_{21}y + s_{22}z, \end{aligned}$$

where

$$(3)\qquad \begin{aligned} r_{11} &= p_{11}^2 + p_{12}p_{21} - p_{11}' - q_{11}, & s_{11} &= p_{11}q_{11} + p_{12}q_{21} - q_{11}', \\ r_{12} &= p_{12}(p_{11}+p_{22}) - p_{12}' - q_{12}, & s_{12} &= p_{11}q_{12} + p_{12}q_{22} - q_{12}', \\ r_{21} &= p_{21}(p_{11}+p_{22}) - p_{21}' - q_{21}, & s_{21} &= p_{21}q_{11} + p_{22}q_{21} - q_{21}', \\ r_{22} &= p_{22}^2 + p_{12}p_{21} - p_{22}' - q_{22}, & s_{22} &= p_{21}q_{12} + p_{22}q_{22} - q_{22}'. \end{aligned}$$

We find by another differentiation

$$(4)\qquad \begin{aligned} y^{(4)} &= l_{11}y' + l_{12}z' + m_{11}y + m_{12}z, \\ z^{(4)} &= l_{21}y' + l_{22}z' + m_{21}y + m_{22}z, \end{aligned}$$

where

$$(5)\qquad \begin{aligned} l_{11} &= -p_{11}r_{11} - p_{21}r_{12} + r_{11}' + s_{11}, & m_{11} &= -r_{11}q_{11} - r_{12}q_{21} + s_{11}', \\ l_{12} &= -p_{12}r_{11} - p_{22}r_{12} + r_{12}' + s_{12}, & m_{12} &= -r_{11}q_{12} - r_{12}q_{22} + s_{12}', \\ l_{21} &= -p_{11}r_{21} - p_{21}r_{22} + r_{21}' + s_{21}, & m_{21} &= -r_{21}q_{11} - r_{22}q_{21} + s_{21}', \\ l_{22} &= -p_{12}r_{21} - p_{22}r_{22} + r_{22}' + s_{22}, & m_{22} &= -r_{21}q_{12} - r_{22}q_{22} + s_{22}'. \end{aligned}$$

If we put further

$$(6)\qquad \Delta_1 = p_{12}s_{12} - q_{12}r_{12}, \quad \Delta_2 = p_{21}s_{21} - q_{21}r_{21},$$

we can find from the above equations:

$$(7)\qquad \begin{aligned} \Delta_1 z &= p_{12}y^{(3)} + r_{12}y'' + (p_{11}r_{12} - p_{12}r_{11})y' + (q_{11}r_{12} - p_{12}s_{11})y, \\ -\Delta_1 z' &= q_{12}y^{(3)} + s_{12}y'' + (p_{11}s_{12} - q_{12}r_{11})y' + (q_{11}s_{12} - q_{12}s_{11})y, \end{aligned}$$

and similarly

$$(8)\qquad \begin{aligned} \Delta_2 y &= p_{21}z^{(3)} + r_{21}z'' + (p_{22}r_{21} - p_{21}r_{22})z' + (q_{22}r_{21} - p_{21}s_{22})z, \\ -\Delta_2 y' &= q_{21}z^{(3)} + s_{21}z'' + (p_{22}s_{21} - q_{21}r_{22})z' + (q_{22}s_{21} - q_{21}s_{22})z. \end{aligned}$$

Finally we obtain the sought-for differential equations for y and z, viz.:

$$(9)\qquad \begin{aligned} \Delta_1 y^{(4)} = {} & (p_{12}m_{12} - q_{12}l_{12})y^{(3)} + (r_{12}m_{12} - s_{12}l_{12})y'' \\ & + [(p_{11}r_{12} - p_{12}r_{11})m_{12} - (p_{11}s_{12} - q_{12}r_{11})l_{12} + \Delta_1 l_{11}]y' \\ & + [(q_{11}r_{12} - p_{12}s_{11})m_{12} - (q_{11}s_{12} - q_{12}s_{11})l_{12} + \Delta_1 m_{11}]y, \end{aligned}$$

and

$$(10)\qquad \begin{aligned} \Delta_2 z^{(4)} = {} & (p_{21}m_{21} - q_{21}l_{21})z^{(3)} + (r_{21}m_{21} - s_{21}l_{21})z'' \\ & + [(p_{22}r_{21} - p_{21}r_{22})m_{21} - (p_{22}s_{21} - q_{21}r_{22})l_{21} + \Delta_2 l_{22}]z' \\ & + [(q_{22}r_{21} - p_{21}s_{22})m_{21} - (q_{22}s_{21} - q_{21}s_{22})l_{21} + \Delta_2 m_{22}]z. \end{aligned}$$

These equations are capable of a vast number of applications. Any question, in fact, in regard to the existence of curves of a specified character on a ruled surface must make use of them.

We notice that the conditions $\Delta_1 = 0$ or $\Delta_2 = 0$ will be necessary and sufficient to make C_y or C_z plane curves; the differential equations (of the third order) of these plane curves are found by putting $\Delta_1 = 0$ or $\Delta_2 = 0$ in (7) or (8) respectively.

We will merely indicate a few other applications of these formulae. Let us write (9) more briefly

$$(9)'\qquad y^{(4)} + 4p_1 y^{(3)} + 6p_2 y'' + 4p_3 y' + p_4 y = 0.$$

We shall find, in the next chapter, the conditions that the integral curve of $(9)'$ may belong to a linear complex, or that it may be a twisted cubic. In one case its invariant of weight 3, and in the other both of its fundamental invariants must vanish. These two conditions, which we now find expressed in terms of the coefficients of (1), will be satisfied only by a particular kind of ruled surface, characterized by the property of containing such curves. One may impose further conditions; for example such that special curves shall be flecnode curves or asymptotic curves on the surface, and then proceed to study the particular class of surface characterized.

It is not our intention to follow up any of these *special* problems, interesting as they are. We shall, however, apply our equations to the problem of answering some questions of a fundamental nature in the *general* theory of ruled surfaces. Some of the special problems just indicated will be considered in the theory of space curves.

§ 2. On ruled surfaces, one of the branches of whose flecnode curve is given.

The flecnode curve is so important in the general theory of ruled surfaces, that it seems essential to investigate to what extent it may be arbitrarily assigned.

If one of the sheets of the flecnode surface, F' of S, is given, there remain only two possibilities for S, namely one or the other of the two sheets of the flecnode surface of F'.

But, let us suppose that we merely know that a certain curve C is one of the branches of the flecnode curve on S. Then there are two questions to answer. Can this curve be chosen arbitrarily? And how far does it determine the surface S?

Let the curve C be given by means of its differential equation

$$\frac{d^4\bar{y}}{d\bar{x}^4} + 4\bar{p}_1\frac{d^3\bar{y}}{d\bar{x}^3} + 6\bar{p}_2\frac{d^2\bar{y}}{d\bar{x}^2} + 4\bar{p}_3\frac{d\bar{y}}{d\bar{x}} + \bar{p}_4\bar{y} = 0, \tag{11}$$

where $p_1, \ldots p_4$ are given functions of x. In the system of differential equations (1) defining our surface S, we must regard the coefficients p_{ik} and q_{ik} as unknown functions. We may, however, assume without exception that $u_{12} = 0$, so that C_u is one of the branches of the flecnode curve on S, that $p_{21} = 0$, so that C_z is an asymptotic curve on S, and that $p_{11} = p_{22} = 0$. Under these assumptions we form the differential equation (9) of the curve C_y. Since C_y is to be identical with C, it must be possible to transform equation (9) into (11) by a transformation of the form

$$y = \varphi(x)\bar{y}, \quad \bar{x} = f(x). \tag{12}$$

The functions φ and f are not independent however. For, while the equations $u_{12}=0$ and $p_{21}=0$ are not disturbed by any transformation of this form, the conditions $p_{11}=p_{22}=0$ are. In fact, a transformation of the form (12) converts (1) into another system of the same form whose corresponding coefficients p_{11} and p_{22} will be

$$\bar{p}_{11}=\frac{p_{11}+2\frac{\varphi'}{\varphi}+\frac{f''}{f'}}{f'},\qquad \bar{p}_{22}=\frac{p_{22}+2\frac{\varphi'}{\varphi}+\frac{f''}{f'}}{f'}.$$

In order, therefore, that after this transformation p_{11} and p_{22} may again vanish, we must have

$$\varphi=\frac{C}{\sqrt{f'}},\tag{13}$$

where C is an arbitrary constant, which may be put equal to unity.

If then we apply the transformation (12) to (11), we shall get an equation

$$\frac{d^4y}{dx^4}+4p_1\frac{d^3y}{dx^3}+6p_2\frac{d^2y}{dx^2}+4p_3\frac{dy}{dx}+p_4y=0,$$

which we must identify with (9). Equating coefficients gives us a system of four equations with five unknown functions of x, viz.: $f, p_{12}, q_{11}, q_{21}, q_{22}$.

We find, therefore, the following theorem: *An arbitrary space curve being given, it can be considered as one branch of the flecnode curve of an infinity of ruled surfaces, into whose general expression there enters an arbitrary function.* One may, therefore, impose another condition and still obtain an infinity of ruled surfaces.

The most general curve C_z which is capable of being the second branch of the flecnode curve on a ruled surface, for which C_y is the first branch, involves therefore, in its expression one arbitrary function of x. It cannot, therefore, be an arbitrary curve, as that would involve two arbitrary functions.

Therefore, *two curves taken at random cannot be connected, point to point, in such a way as to constitute the complete flecnode curve upon the ruled surface thus generated.*

We may also prove our theorem by purely synthetic considerations. Let us take points $P_1, P_2, P_3, P_4, \ldots$ on an arbitrary curve, corresponding for example to equal increments $\varDelta x$ of the parameter. Through P_1, P_2, P_3 draw three arbitrary lines g_1, g_2, g_3. We can draw a line f_1 through P_1 intersecting g_2 and g_3, say in Q_2 and Q_3. Take an arbitrary point Q_4 on f_1, and join it to P_4 by a line g_4. Then f_1 intersects g_1, g_2, g_3, g_4. Through P_2 we draw a line f_2 intersecting g_3 and g_4 in points Q_3', Q_4' and of course g_2 in $Q_2'=P_2$. Take an arbitrary point Q_5' on f_2 and join it to P_5 by a line g_5. Continue this process. Clearly, we shall get two assemblages of lines

$g_1, g_2, \ldots$ and $f_1, f_2, \ldots$, which, when $P_1, P_2, \ldots$ are taken closer and closer together, approach as limits two ruled surfaces having the given curve as flecnode curve, and which are flecnode surfaces of each other. The first three lines g_1, g_2, g_3 are arbitrary, which gives rise to six constants of integration. Further, the double ratios (P_1, Q_2, Q_3, Q_4), (Q_2', Q_3', Q_4', Q_5') etc. may be chosen arbitrarily which brings into evidence the arbitrary function involved in the construction of these surfaces.

The construction, which has just been described, becomes indeterminate if the given curve C is a straight line. For then Q_4 coincides with P_4, etc. In fact, the most general ruled surface with a given straight line directrix depends on two arbitrary functions.

If the given curve C is to be at the same time the second branch of the flecnode curve, i. e. if both of the branches of the flecnode curve of S coincide with C, g_4 must be tangent to the hyperboloid determined by g_1, g_2, g_3; g_5 must be tangent to the hyperboloid determined by g_2, g_3, g_4; etc. This condition, therefore, clearly fixes the double ratios $(Q_1 Q_2 Q_3 Q_4)$, etc., i e. the arbitrary function. Therefore, this problem has in general ∞^6 solutions.

Let us assume that C_y is not a straight line. Let us call the developable surface formed by the tangents of C_y its *primary* developable. There exists another important developable surface containing C_y, which we shall speak of as its *secondary* developable, as indicated in the following theorem

1. *If at every point of the flecnode curve of S there be drawn the generator of the surface, the flecnode tangent, the tangent of the flecnode curve, and finally the line which is the harmonic conjugate of the latter with respect to the other two, the locus of these last lines is a developable surface, the secondary developable of the flecnode curve.*

2. *We can find a single infinity of ruled surfaces, each having one branch of its flecnode curve in common with that of S. This family of ∞^1 surfaces may be described as an involution, of which any surface of the family and its flecnode surface form a pair. The primary and secondary developables of the branch of the flecnode surface considered, are the double surfaces of this involution. In fact, the generators of these surfaces, at every point of their common flecnode curve, form an involution in the usual sense*

We proceed to prove these theorems. Since C_y is a branch of the flecnode curve, we may assume $u_{12} = p_{11} = p_{22} = 0$. System (1) assumes the form

$$(14) \quad y'' + p_{12} z' + q_{11} y + \frac{1}{2} p_{12}' z = 0, \quad z'' + p_{21} y' + q_{21} y + q_{22} z = 0.$$

The flecnode tangent at P_y is the line joining P_y to P_ρ, where

$$\varrho = 2y' + p_{12} z,$$

while the tangent of the flecnode curve joins P_y to $P_{y'}$. In the plane pencil formed by these lines, the harmonic conjugate of $P_y P_{y'}$ with respect to $P_y P_z$ and $P_y P_\varrho$ will be the line $P_y P_\tau$, where

$$\tau = y' + p_{12} z.$$

But from the first equation of (14) we find at once

$$\tau' + q_{11} y - \frac{1}{2} \frac{p_{12}'}{p_{12}} (\tau - y') = 0, \tag{15}$$

i. e. $P_y P_\tau$ generates a *developable* surface as asserted in the first theorem.

Put

$$\begin{aligned} e &= \tau + k y' = (1 + k) y' + p_{12} z, \\ f &= \tau - k y' = (1 - k) y' + p_{12} z, \end{aligned} \tag{16}$$

where k is a constant. Clearly the lines $P_y P_e$ and $P_y P_f$ form a pair of the involution whose double lines are $P_y P_{y'}$ and $P_y P_\tau$.

One finds that y and e satisfy the following system of differential equations:

$$\begin{aligned} y'' + P_{11} y' + P_{12} e' + Q_{11} y + Q_{12} e &= 0, \\ e'' + P_{21} y' + P_{22} e' + Q_{21} y + Q_{22} e &= 0, \end{aligned} \tag{17}$$

where

$$\begin{aligned}
&P_{11} = -\frac{1+k}{2k} \frac{p_{12}'}{p_{12}}, \quad P_{12} = -\frac{1}{k}, \quad Q_{11} = -\frac{q_{11}}{k}, \quad Q_{12} = \frac{1}{2k} \frac{p_{12}'}{p_{12}}, \\
&P_{21} = (1+k) q_{11} + k(1+k) q_{22} - k p_{12} p_{21} \\
&\qquad + \frac{(1-3k)(1-k^2)}{4k} \left(\frac{p_{12}'}{p_{12}}\right)^2 + \frac{1-k}{2} \frac{p_{12}''}{p_{12}}, \\
&P_{22} = \frac{1-3k}{2k} \frac{p_{12}'}{p_{12}}, \\
&Q_{21} = (1+k) q_{11}' - k p_{12} q_{21} + \frac{(1-3k)(1+k)}{2k} \frac{p_{12}'}{p_{12}} q_{11}, \\
&Q_{22} = -\left[k q_{12} + \frac{(1-3k)(1-k)}{4k} \left(\frac{p_{12}'}{p_{12}}\right)^2 + \frac{1-k}{2} \frac{p_{12}''}{p_{12}}\right].
\end{aligned} \tag{18}$$

We find

$$C_{12} = 2P_{12}' - 4Q_{12} + P_{12}(P_{11} + P_{22}) = 0,$$

i. e. the curve C_y is flecnode curve on the ruled surface S_k generated by $P_y P_e$. The flecnode surface of S_k is obtained by joining P_y to the point

$$2y' + P_{11} y + P_{12} e = -\frac{1}{k} f - \frac{1+k}{2k} \frac{p_{12}'}{p_{12}} y,$$

a point on the line $P_y P_f$. We see, therefore, that the ruled surfaces S_k and S_{-k} are flecnode surfaces of each other. We have now proved

our second theorem, and we may speak of an involution of ruled surfaces having one branch of their flecnode curve in common. The double surfaces of the involution are developables, while the members of each pair of the involution are flecnode surfaces of each other.

We have seen that $P_y P_\tau$ generates a developable. If

$$g = \alpha y + \beta \tau$$

represents its edge of regression, it must be possible to represent g' in the form

$$g' = \gamma y + \delta \tau,$$

since the line $P_y P_\tau$ must then be tangent to the curve C_g.

We find, by differentiation, making use of (15),

$$g' = \alpha y' + \beta \left(\frac{1}{2} \frac{p_{12}'}{p_{12}} \tau - q_{11} y - \frac{1}{2} \frac{p_{12}'}{p_{12}} y' \right) + \alpha' y + \beta' \tau,$$

so that g' will be of the required form, if and only if

$$\alpha : \beta = p_{12}' : 2p_{12}.$$

Therefore

$$g = p_{12}' y + 2p_{12} \tau.$$

If we express τ in terms of y, z and ϱ, we shall find

$$g = p_{12} \varrho + p_{12}' y + p_{12}^2 z, \tag{19a}$$

as the expression for the edge of regression of the secondary developable of the branch C_y of the flecnode curve. Similarly, if $\Theta_1 \neq 0$,

$$h = p_{21} \sigma + p_{21}' z + p_{21}^2 y \tag{19b}$$

will represent the cuspidal edge of the secondary developable of the branch C_z of the flecnode curve, assuming of course $u_{21} = 0$.

One easily finds

$$-\frac{3}{2} p_{12}' g + p_{12} g' = \lambda y, \qquad \frac{3}{2} p_{21}' h + p_{21} h' = \mu z, \tag{20}$$

where

$$\begin{aligned} \lambda &= p_{12} p_{12}'' + \frac{1}{2} p_{12}^2 u_{11} - p_{12}^3 p_{21} - \frac{3}{2} (p_{12}')^2, \\ \mu &= p_{21} p_{21}'' + \frac{1}{2} p_{21}^2 u_{22} - p_{21}^3 p_{12} - \frac{3}{2} (p_{21}')^2. \end{aligned} \tag{21}$$

The system of differential equations, of which g and h are the solutions, has the coefficients

$$\begin{aligned} P_{11} &= -\frac{1}{\lambda} \left[\lambda' + \frac{3}{4} p_{12}^2 p_{12}' p_{21} \right], \quad P_{12} = \frac{p_{21} \lambda}{\mu}, \\ Q_{11} &= -\frac{1}{\lambda p_{12}} \Big[\lambda \left(2p_{12}'' + \frac{1}{4} p_{12} u_{11} - \frac{1}{2} p_{12}^2 p_{21} \right) \\ &\qquad - \frac{3}{2} p_{12}' \left(\lambda' + \frac{3}{4} p_{12}^2 p_{12}' p_{21} \right) \Big], \\ Q_{12} &= -\frac{3 p_{21}' \lambda}{2\mu}, \end{aligned} \tag{22}$$

while P_{21}, P_{22}, Q_{21}, Q_{22} are obtained from these same equations by permuting the indices 1 and 2, and consequently also the letters λ and μ.

We see that we obtain, in this way, corresponding uniquely to any ruled surface, whose flecnode curve intersects every generator in two distinct points, another ruled surface, which is generated by the lines joining corresponding points of the edges of regression of the secondary developables of the two branches of the flecnode curve.

Equations (20) show that one of the secondary developables of C_y and C_z degenerates into a cone if λ or μ vanishes. In that case our new ruled surface also becomes a cone. If both of the secondary developables are cones, this ruled surface degenerates into the straight line joining their vertices.

Equations (22) show that this new ruled surface cannot be developable except if λ or μ is zero, i. e. unless it is a cone. For the possibility $p_{12} = 0$ or $p_{21} = 0$ is to be excluded, since we should then have a ruled surface S with a straight line directrix.

§ 3. On ruled surfaces one of the branches of whose complex curve is given.

There exists an infinity of ruled surfaces, each of which contain an arbitrarily given curve as one branch of its complex curve. Into the general analytical expression of these surfaces there enters an arbitrary function.

The analytical proof for this statement is precisely similar to that of the corresponding theorem of § 2 We shall give at once a geometrical construction for these surfaces.

Let us consider five straight lines $g_1, \ldots g_5$ Let f_1', f_1'' be the two transversals of $g_1, \ldots g_4$, and f_2', f_2'' those of $g_2, \ldots g_5$. Clearly $g_1, \ldots g_5$ determine a linear complex, with respect to which f_1', f_1'' and f_2', f_2'' are two pairs of reciprocal polars. Take a point P on g_1. The plane, which corresponds to it in the linear complex, passes through g_1 and the line h_1 which passes through P and intersects both f_2' and f_2''. If $g_1, \ldots g_5$ are made to approach each other, we shall have, in the limit, five consecutive generators of a ruled surface and its osculating linear complex. The plane tangent to this ruled surface at P is the limit of the plane containing g_1 and the line through P which intersects g_2 and g_3, i. e. the asymptotic tangent of the surface at P. If P is a point on the complex curve, h_1 must be in the plane tangent to the ruled surface at P

Now let an arbitrary curve be given, and let us choose points upon it, P_1, P_2, $P_3, \ldots$ according to any law. Through $P_1, \ldots P_4$ draw four arbitrary lines $g_1, \ldots g_4$. Through P_1 draw h_1, the line which intersects g_2 and g_3. In the plane of g_1 and k_1 draw any

line h_1 through P_1. The line g_5, through P_5, is to be constructed in such a way that the two transversals of $g_2, \ldots g_5$ shall both meet h_1. Now these transversals must be generators of the second set on the hyperboloid determined by g_2, g_3, g_4. They must, therefore, be those two generators of the second set, f_2' and f_2'', which pass through the two points in which h_1 intersects the hyperboloid. There exists just one line through P_5 intersecting both f_2' and f_2''. It is the line g_5. In the same way, starting with $g_2, \ldots g_5$, we can construct g_6, etc. Finally we pass to the limit. There enters an arbitrary function, fixing the position of the successive lines $h_1, h_2, \ldots$ in the planes in which they must lie.

We may easily solve the problem; to determine all ruled surfaces, an asymptotic curve of which is given. In fact, if C_y is the given asymptotic curve, any other curve of the ruled surface will be given by the expression

$$z = \alpha y + \beta y' + \gamma y''.$$

The equations of this chapter enable us to write down the special form which z must have, so that C_z may also be an asymptotic curve. We shall then have the ruled surface referred to its asymptotic curves in an explicit form; by restricting the functions $y_1, \ldots y_4$, etc to algebraic values, we shall thus find the most general ruled surface, all of whose asymptotic lines are algebraic. It is a mere application of our general equations to deduce these results, which were first obtained by *Koenigs*[1]) in 1888. This paper of *Koenigs* is remarkable also in so far, as it seems to be the only one in the litterature of the theory of ruled surfaces, which makes use of a system of differential equations of the form (A). But even here, no stress is laid upon this fact, and no further consequences are drawn therefrom. The system is used merely as an auxiliary, its fundamental importance for the theory of ruled surface not being recognized.

Examples.

Ex. 1. Express the condition, that one or both of the branches of the flecnode curve may be plane curves, in terms of the invariants.

Ex. 2.* What are the conditions under which one or both branches of the flecnode curve may be conics? space cubics?

Ex. 3. Let C_y be any curve on S. At the point P_y of C_y construct the harmonic conjugate of the tangent to C_y with respect to the generator and the other asymptotic tangent of P_y. Prove that these lines generate a developable, and find its cuspidal edge (cf. Chapter IX, Ex. 4).

1) *Koenigs*. Détermination sous forme explicite de toute surface reglée rapportée à ses lignes asymptotiques, et en particulier de toutes les surfaces reglées à lignes asymptotiques algébriques. *Comptes Rendus*, vol. 106 (1888) p. 51—54.

CHAPTER XIII.

PROJECTIVE DIFFERENTIAL GEOMETRY OF SPACE CURVES

§ 1. The invariants and covariants for $n = 4$.

It was shown in Chapter II that the differential geometry of a space curve could be based upon the consideration of the linear homogeneous differential equation of the fourth order

$$(1) \qquad y^{(4)} + 4p_1 y^{(3)} + 6p_2 y'' + 4p_3 y' + p_4 y = 0.$$

The invariants and covariants of a linear homogeneous differential equation of the n^{th} order have been computed, in their canonical form, for every value of n. It suffices, therefore, to put $n = 4$ in the equations of Chapter II, in order to obtain the canonical expressions for these functions. But we shall need the un-canonical form of the invariants for our more detailed discussion of the case $n = 4$. It becomes necessary, therefore, to write down explicitly a number of equations, which are really included as special cases in the equations of Chapter II. At the same time we obtain, in this way, a verification of the general theorems of that chapter for this special case, thus making the theory of space curves independent of that general theory, at the cost of some repetition.

If we make the transformation

$$y = \lambda(x)\bar{y},$$

where λ is an arbitrary function of x, we shall find for $\bar{y}$ an equation of the same form as (1),

$$\bar{y}^{(4)} + 4\pi_1 \bar{y}^{(3)} + 6\pi_2 \bar{y}'' + 4\pi_3 \bar{y}' + \pi_4 \bar{y} = 0,$$

where

$$(2) \qquad \begin{aligned} \pi_1 &= \frac{\lambda' + p_1\lambda}{\lambda}, \\ \pi_2 &= \frac{\lambda'' + 2p_1\lambda' + p_2\lambda}{\lambda}, \\ \pi_3 &= \frac{\lambda^{(3)} + 3p_1\lambda'' + 3p_2\lambda' + p_3\lambda}{\lambda}, \\ \pi_4 &= \frac{\lambda^{(4)} + 4p_1\lambda^{(3)} + 6p_2\lambda'' + 4p_3\lambda' + p_4\lambda}{\lambda}, \end{aligned}$$

whence one may deduce the absolute seminvariants

$$
\begin{aligned}
&P_2 = p_2 - p_1' - p_1^2,\\
(3)\qquad &P_3 = p_3 - p_1'' - 3p_1p_2 + 2p_1^3,\\
&P_4 = p_4 - 4p_1p_3 - 3p_2^2 + 12p_1^2p_2 - 6p_1^4 - p_1^{(3)},
\end{aligned}
$$

and the relative semi-covariants, besides y, which is obviously itself a semi-covariant,

$$
\begin{aligned}
z &= y' + p_1y,\\
\varrho &= y'' + 2p_1y' + p_2y,\\
\sigma &= y^{(3)} + 3p_1y'' + 3p_2y' + p_3y.
\end{aligned}
$$

The absolute semi-covariants are $\frac{z}{y}, \frac{\varrho}{y}, \frac{\sigma}{y}$. All other semi-covariants and seminvariants are functions of these and of the derivatives of P_2, P_3, P_4.

From (4) we deduce the following equations, which we shall use later:

$$
\begin{aligned}
&y' = -p_1y + z,\\
&z' = -P_2y - p_1z + \varrho,\\
(5)\qquad &\varrho' = -(P_3 - P_2')y - 2P_2z - p_1\varrho + \sigma,\\
&\sigma' = -(P_4 - P_3')y - 3(P_3 - P_2')z - 3P_2\varrho - p_1\sigma,
\end{aligned}
$$

and also

$$
\begin{aligned}
&y'' = (2p_1^2 - p_2)y - 2p_1z + \varrho,\\
&y^{(3)} = (-p_3 + 6p_1p_2 - 6p_1^3)y + (-3p_2 + 6p_1^2)z - 3p_1\varrho + \sigma,\\
(6)\qquad &y^{(4)} = (-p_4 + 8p_1p_3 - 36p_1^2p_2 + 6p_2^2 + 24p_1^4)y\\
&\qquad + (-4p_3 + 24p_1p_2 - 24p_1^3)z + (-6p_2 + 12p_1^2)\varrho - 4p_1\sigma.
\end{aligned}
$$

We now proceed to make a transformation of the independent variable $\xi = \xi(x)$. We find, denoting the coefficients of the transformed equation by p_k,

$$
\begin{aligned}
&p_1 = \frac{1}{\xi'}\left(p_1 + \frac{3}{2}\eta\right),\\
&p_2 = \frac{1}{(\xi')^2}\left[p_2 + 2\eta p_1 + \frac{1}{6}(4\mu + 9\eta^2)\right],\\
(7)\qquad &p_3 = \frac{1}{(\xi')^3}\left[p_3 + \frac{3}{2}\eta p_2 + \left(\mu + \frac{3}{2}\eta^2\right)p_1 + \frac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)\right],\\
&p_4 = \frac{1}{(\xi')^4}p_4,
\end{aligned}
$$

where we have put

$$
(8)\qquad \eta = \frac{\xi''}{\xi'}, \quad \mu = \eta' - \frac{1}{2}\eta^2.
$$

We find further

$$\bar{z} = \frac{1}{\xi'}\left(z + \frac{3}{2}\eta y\right),$$

(9) $$\bar{\varrho} = \frac{1}{(\xi')^2}\left[\varrho + 2\eta z + \frac{1}{6}(4\mu + 9\eta^2)y\right],$$

$$\bar{\sigma} = \frac{1}{(\xi')^3}\left[\sigma + \frac{3}{2}\eta\varrho + \left(\mu + \frac{3}{2}\eta^2\right)z + \frac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)y\right],$$

so that $\frac{z}{y}, \frac{\varrho}{y}, \frac{\sigma}{y}$ are cogredient with p_1, p_2, p_3.

Making use of these equations, we find

$$P_2 = \frac{1}{(\xi')^2}\left[P_2 - \frac{5}{6}\mu\right],$$

(10) $$P_3 = \frac{1}{(\xi')^3}\left[P_3 - 3\eta P_2 - \frac{5}{4}\mu' + \frac{5}{2}\eta\mu\right],$$

$$P_4 = \frac{1}{(\xi')^4}\Big[P_4 - 6\eta P_3 - 4\eta' P_2 + 11\eta^2 P_2$$
$$+ \frac{15}{2}\eta\mu' - \frac{15}{2}\eta^2\mu \quad \frac{3}{2}\mu'' + \frac{19}{6}\mu^2\Big],$$

whence

$$P_2' = \frac{1}{(\xi')^3}\left[P_2' - 2\eta P_2 - \frac{5}{6}\mu' + \frac{5}{3}\eta\mu\right],$$

$$P_2'' = \frac{1}{(\xi')^4}\Big[P_2'' - 5\eta P_2' - 2\mu P_2 + 5\eta^2 P_2 - \frac{5}{6}\mu''$$

(11) $$+ \frac{5}{3}\mu^2 - \frac{25}{6}\mu\eta^2 + \frac{25}{6}\mu'\eta\Big],$$

$$P_3' = \frac{1}{(\xi')^4}\Big[P_3' - 3\eta(P_2' + P_3) - 3\mu P_2 + \frac{15}{2}\eta^2 P_2 - \frac{5}{4}\mu''$$
$$+ \frac{25}{4}\eta\mu' \quad \frac{25}{4}\mu\eta^2 + \frac{5}{2}\mu^2\Big].$$

We find, therefore, the following invariants and covariants

$$\Theta_3 = P_3 - \frac{3}{2}P_2', \quad \Theta_4 = P_4 - 2P_3' + \frac{6}{5}P_2'' \quad \frac{6}{25}P_2^2,$$

$$\Theta_{8\,1} = 6\Theta_3\Theta_3'' - 7\Theta_3'^2 - \frac{108}{5}P_2\Theta_3^2,$$

(12) $$C_2 = 10z^2 - 15y\varrho - 12P_2y^2,$$

$$C_3 = 10z^3 - 3C_2z - 9(5\sigma + 6P_2z + P_3y)y^2,$$

$$C_4 = 2\Theta_3 z + \Theta_3' y,$$

where the index indicates the weight. In denoting one invariant of weight 8 by $\Theta_{8\,1}$, we follow the general notation explained in Chapter II equ. (54). An invariant may be regarded as a covariant of degree zero. With this understanding, it suffices to say that the effect of the complete transformation

$$\bar{y} = \lambda(x) y, \quad \xi = \xi(x),$$

upon a covariant of degree d and of weight w, is to transform it into $\bar{C}$, where

$$\bar{C} = \frac{\lambda^d}{(\xi')^w} C.$$

The general theory shows that all other invariants and covariants may be deduced from these by algebraic and differentiation processes.

§ 2. Canonical forms.

Equations (2) show that, if we make the transformation

$$y = e^{-\int p_1 dx} \bar{y},$$

the coefficients of the resulting equation for $\bar{y}$ will be

$$\pi_1 = 0, \quad \pi_2 = P_2, \quad \pi_3 = P_3, \quad \pi_4 = P_4 + 3 P_2^2.$$

We shall say that the equation has been put into the *semi-canonical* form.

From (10) we see that if $\xi(x)$ be chosen so that

$$\eta' - \frac{1}{2} \eta^2 = \mu = \frac{6}{5} P_2, \tag{13}$$

in the resulting equation P_2 will be zero. Since P_2 is a seminvariant, any transformation of the form

$$\bar{y} = \lambda y$$

will not disturb the equation $P_2 = 0$, and we may again choose λ so as to make the coefficient of $\frac{d^3 \bar{y}}{d\xi^3}$ vanish. It is, therefore, always possible to reduce the equation to the form

$$\frac{d^4 \bar{y}}{d\xi^4} + 4\pi_3 \frac{d\bar{y}}{d\xi} + \pi_4 y = 0,$$

which we shall call the *Laguerre-Forsyth canonical form*. This is equivalent to assuming $p_1 = p_2 = 0$ in the original equation.

If $\Theta_3 \neq 0$, we may transform the independent variable so as to make $\Theta_3 = 1$. In fact, we have for an arbitrary transformation

$$\bar{\Theta}_3 = \frac{1}{(\xi')^3} \Theta_3.$$

If, therefore, we put

$$\xi = \int \sqrt[3]{\Theta_3}\, dx, \tag{14}$$

$\bar{\Theta}_3$ will be equal to unity. We may again, by a transformation of the form $\bar{y} = \lambda y$, make p_1 vanish. The canonical form, which is characterized by the conditions

$$p_1 = 0, \ \Theta_3 = 1,$$

we may properly denote as the *Halphen canonical form.*

In our geometrical discussions, only the quantity

$$\eta = \frac{\xi''}{\xi'},$$

not ξ itself will be of importance. λ also is an unimportant factor which has no geometrical significance. Equation (13) shows, therefore, that the reduction to the *Laguerre-Forsyth* form can always be accomplished in ∞^1 essentially different ways. It is important to remark that (13) is an equation of the *Riccati* form, so that the cross-ratio of any four solutions is constant.

The *Halphen* form, on the other hand, can be obtained in just one way, if it exists at all, i. e. if $\Theta_3 \neq 0$. If Θ_3 vanishes, Θ_4 may be reduced to unity unless it also is equal to zero. The case when both Θ_3 and Θ_4 vanish, is especially simple. The *Laguerre-Forsyth* form reduces to

$$\frac{d^4\bar{y}}{d\xi^4} = 0.$$

If two equations of the form (1) can be transformed into each other by a transformation of the kind here considered, we shall call them equivalent. Clearly, for equivalent equations, the corresponding absolute invariants are equal.

If equation (1) is given, the invariants Θ_3, Θ_4, $\Theta_{3\cdot1}$, etc. are known functions of x. Conversely, equations (12) show that if Θ_3, Θ_4, $\Theta_{3\cdot1}$ are given as arbitrary functions of x, provided that $\Theta_3 \neq 0$, P_2, P_3 and P_4 are determined uniquely. If $\Theta_3 = 0$, then $\Theta_{3\cdot1} = 0$ also, and we must assign a further condition. The function

$$(15) \qquad \Theta_{4\cdot1} = 8\,\Theta_4\Theta_4'' - 9(\Theta_4')^2 - \frac{80}{3}\,P_2\Theta_4^2$$

is also an invariant. If $\Theta_3 = 0$, and Θ_4, $\Theta_{4\cdot1}$ are given, P_2, P_3, P_4, are determined uniquely. If both Θ_3 and Θ_4 vanish, all invariants are zero, and the equation may be reduced to the form

$$\frac{d^4\eta}{d\xi^4} = 0.$$

As we may always assume that $p_1 = 0$, we see that *the differential equation* (1) *is essentially determined when its invariants are given as functions of x.*

The *Lagrange adjoint* of (1) is[1])

$$(16) \qquad \begin{aligned} u^{(4)} - 4p_1u^{(3)} + 6(p_2 - 2p_1')u'' - 4(p_3 - 3p_2' + 3p_1'')u' \\ + (p_4 - 4p_3' + 6p_2'' - 4p^{(3)})u = 0. \end{aligned}$$

If $y_1, \ldots y_4$ constitute a fundamental system of (1), the minors of $x_1, \ldots x_4$ in the determinant

1) Cf. Chapter II, § 5.

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ y_1', & y_2', & y_3', & y_4' \\ y_1'', & y_2'', & y_3'', & y_4'' \end{vmatrix},$$

multiplied by a common factor, which does not interest us, form a fundamental system of (16).

If we denote the seminvariants of (16) by Π_2, Π_3, Π_4, we have

$$(17)\quad \Pi_2 = P_2, \quad \Pi_3 = -P_3 + 3P_2', \quad \Pi_4 = P_4 - 4P_3' + 6P_2'',$$

whence follows reciprocally

$$P_2 = \Pi_2, \quad P_3 = -\Pi_3 + 3\Pi_2', \quad P_4 = \Pi_4 - 4\Pi_3' + 6\Pi_2''.$$

The invariants of (16) differ from the invariants of (1) only in this that the sign of Θ_3 is changed.

§ 3. Geometrical Interpretation.

If the functions $y_1, \ldots y_4$ constitute a fundamental system of (1), we may interpret them as the homogeneous coordinates of a point P_y of a curve C_y in ordinary space. The coefficients p_1, p_2, p_3, p_4 of (1) are invariants of the general projective group. The transformation

$$y = \lambda \bar{y}$$

does not change the ratios $y_1 : y_2 : y_3 : y_4$, and therefore leaves the curve C_y invariant. The transformation $\xi = \xi(x)$ merely changes the parameter in terms of which the coordinates are expressed. It is clear, therefore, that any system of equations, invariant under these transformations, expresses a projective property of the curve C_y.

The *Lagrange adjoint* of (1) may be taken to represent the same curve in tangential coordinates, or else a reciprocal curve in point coordinates.

We may, therefore, state the results of § 2 as follows. *If the invariants of a curve are given as functions of x, the curve is determined except for projective transformations. If the invariants of two curves, except those of weight three, are respectively equal to each other, while the invariants of weight three differ only in sign, the two curves are dualistic to each other. Those curves are self-dual for which $\Theta_3 = 0$.* Moreover, these latter curves are the only curves which are identically self-dual; i. e. for which a dualistic transformation exists which converts every point of the curve into the osculating plane of that point, and vice versa, while every tangent is transformed into itself.

If we put $y = y_k$ $(k = 1, 2, 3, 4)$ into the expressions for z, ϱ, σ we obtain three other points P_z, P_ϱ, P_σ which, as x varies, describe

curves C_z, C_ϱ, C_σ, curves which are closely connected with C_y. P_z is clearly a point on the tangent of C_y constructed at P_y; P_ϱ is in the plane osculating C_y at P_y, while P_σ is outside of this plane. These four points are never coplanar except at those exceptional points of C_y, whose osculating planes are stationary, i. e. have more than three consecutive points in common with the curve.

In order to study the curve C_y in the vicinity of P_y, it will, therefore, be convenient to introduce the tetrahedron $P_y P_z P_\varrho P_\sigma$ as tetrahedron of reference, with the further convention that, if any expressions of the form

$$u_k = \alpha_1 y_k + \alpha_2 z_k + \alpha_3 \varrho_k + \alpha_4 \sigma_k \ (k = 1, 2, 3, 4)$$

present themselves, the coordinates of the corresponding point P_u shall be

$$(\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4).$$

In writing u_k the index k may be suppressed, so that the single expression

$$\alpha_1 y + \alpha_2 z + \alpha_3 \varrho + \alpha_4 \sigma$$

represents the point $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ adequately.

If the independent variable x is transformed, the tetrahedron of reference is changed in accordance with equations (9). P_y of course remains the same; P_z is changed into $P_{\bar{z}}$, which may obviously be any point on the tangent; etc. Thus, while an arbitrary transformation of the parameter x does not affect the curve C_y itself, it does very materially affect the semi-covariant curves C_z, C_ϱ, C_σ. It is clear, however, that two transformations of x, which give rise to the same η, are geometrically equivalent. We may also, without affecting the position of the points P_z, P_ϱ, P_σ, assume that (1) is written in the semi-canonical form, so that $p_1 = 0$. For, in order to put (1) into the semi-canonical form, we need only multiply y by a certain factor λ, which will then also appear multiplied into the semi-covariants z, ϱ and σ.

Let us then assume $p_1 = 0$. We shall have $z = y'$. If we differentiate (1) and eliminate y between the resulting equation and (1), we shall find

$$\begin{aligned}(18)\qquad &(P_4 + 3P_2^2)z^{(4)} - (P_4' + 6P_2P_2')z^{(3)} + 6P_2(P_4 + 3P_2^2)z'' \\ &+ [(6P_2' + 4P_3)(P_4 + 3P_2^2) - 6P_2(P_4' + 6P_2P_2')]z' \\ &+ [(4P_3' + P_4 + 3P_2^2)(P_4 + 3P_2^2) - 4P_3(P_4' + 6P_2P_2')]z = 0,\end{aligned}$$

if $P_4 + 3P_2^2 \neq 0$. If $P_4 + 3P_2^2 = 0$ we find

$$(19)\qquad z^{(3)} + 6P_2 z' + 4P_3 z = 0.$$

Equation (18) determines the curve C_z in the same way as (1) determines C_y. But if $P_4 + 3P_2^2 = 0$, z satisfies (19) showing that

the curve C_z is in this case a plane curve. Therefore, if the variable ξ be so chosen as to make $\bar{P}_4 + 3\bar{P}_2^2 = 0$, the corresponding curve C_z is a plane section of the developable surface whose cuspidal edge is C_y. In harmony with this, equations (10) show that the most general value of η, which satisfies the condition $\bar{P}_4 + 3\bar{P}_2^2 = 0$, contains three arbitrary constants, as it should since there are ∞^3 planes in space.

We shall need to consider the ruled surfaces generated by those edges of our tetrahedron which meet in P_y. Of these we know one immediately, namely the developable which has C_y as its edge of regression, and of which P_yP_z is a generator. The ruled surface generated by P_yP_ϱ clearly has C_y as an asymptotic curve; for, the plane $P_yP_\varrho P_z$ is both osculating plane of C_y at P_y, and tangent plane of the surface at P_y. If we assume $p_1 = 0$, this ruled surface may be studied by means of the equations

$$(20)\qquad \begin{aligned} &y'' + P_2 y - \varrho = 0,\\ &\varrho'' + (4P_3 - 2P_2')y' + (P_4 - P_2'' - 2P_2^2)y + 5P_2\varrho = 0, \end{aligned}$$

in accordance with the general theory of ruled surfaces as developed in the preceding chapters of this book. To prove (20) we need only differentiate the expression for ϱ twice, express $y^{(3)}$ and $y^{(4)}$ in terms of y, z, ϱ, σ, and eliminate z and σ.

The ruled surface generated by P_yP_σ is especially important. We have

$$(21)\qquad \sigma = y^{(3)} + 3p_1y^{(2)} + 3p_2y' + p_3y,$$

whence

$$(22)\qquad \begin{aligned} \sigma' &= y^{(4)} + 3p_1y^{(3)} + (3p_1' + 3p_2)y'' + (3p_2' + p_3)y' + p_3'y,\\ \sigma'' &= y^{(5)} + 3p_1y^{(4)} + 3(2p_1' + p_2)y^{(3)} + (3p_1'' + 6p_2' + p_3)y''\\ &\qquad + (3p_2'' + 2p_3')y' + p_3''y. \end{aligned}$$

From (21) we find

$$(23)\qquad \begin{aligned} y^{(3)} &= \sigma - 3p_1y'' - 3p_2y' - p_3y,\\ y^{(4)} &= \sigma' - 3p_1\sigma - 3(p_1' + p_2 - 3p_1^2)y'' - (3p_2' + p_3 - 9p_1p_2)y'\\ &\qquad - (p_3' - 3p_1p_3)y. \end{aligned}$$

If we substitute these values in (1), we obtain the equation

$$(24)\qquad \begin{aligned} &3(p_2 - p_1' - p_1^2)y'' + 3(p_3 - p_2' - p_1p_2)y' + \sigma'\\ &\qquad + (p_4 - p_1p_3 - p_3')y + p_1\sigma = 0, \end{aligned}$$

where the coefficient of y'' is $3P_2$.

Let us differentiate both members of this equation, and eliminate y'' and $y^{(3)}$ by means of (23) and (24). We shall find

$$
\begin{aligned}
3P_2\sigma'' = (q_3 r_3 + 3P_2 q_4)y' + (3P_2 q_1 - q_3)\sigma' + (r_4 q_3 + 3P_2 q_5)y \\
+ (3P_2 q_2 - p_1 q_3)\sigma,
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
& q_1 = -p_1, \quad q_2 = 2p_1' - 3p_2 + 3p_1^2, \quad q_3 = -3P_3 + 3p_1 P_2, \\
& q_4 = -2p_3' + 3p_2'' + 3p_1 p_2' - 6p_1' p_2 - p_4 + p_1 p_3 + 9p_2^2 - 9p_1^2 p_2, \\
& q_5 = -(p_4' - p_3'' - p_1 p_3' - p_1' p_3) + 3p_3(p_2 - p_1' - p_1^2), \\
& r_3 = -3(p_3 - p_2' - p_1 p_2), \\
& r_4 = -(p_4 - p_3' - p_1 p_3).
\end{aligned} \tag{26}
$$

Equations (24) and (25) define the ruled surface generated by $P_y P_\sigma$. If we assume $p_1 = 0$, we find

$$
\begin{aligned}
y'' + p_{11} y' + p_{12}\sigma' + q_{11} y + q_{12}\sigma = 0, \\
\sigma'' + p_{21} y' + p_{22}\sigma' + q_{21} y + q_{22}\sigma = 0,
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
& p_{11} = \frac{P_3 - P_2'}{P_2}, \quad p_{12} = \frac{1}{3P_2}, \quad q_{11} = \frac{P_4 + 3P_2^2 - P_3'}{3P_2}, \quad q_{12} = 0, \\
& p_{21} = \frac{1}{P_2}[-3P_3^2 + 3P_3 P_2' + 2P_2 P_3' - 3P_2 P_2'' + P_2 P_4 - 6P_2^3], \\
& \qquad p_{22} = -\frac{P_3}{P_2}, \\
& q_{21} = \frac{1}{P_2}[-P_3 P_4 - 6P_2^2 P_3 + P_3 P_3' + P_2 P_4' + 6P_2^2 P_2' - P_2 P_3''], \\
& \qquad q_{22} = 3P_2.
\end{aligned} \tag{28}
$$

If (1) is written in the *Laguerre-Forsyth* form, $P_2 = 0$. In that case, the two equations (27) reduce to the single equation

$$
P_3 y' + \frac{1}{3}\sigma' + \frac{1}{3}(P_4 - P_3')y = 0, \tag{29}
$$

which proves that, *in this case, the surface generated by $P_y P_\sigma$ is developable.* For, the tangents constructed to C_y at P_y and to C_σ at P_σ are then coplanar.[1]) Moreover, only if $P_2 = 0$ will the surface generated by $P_y P_\sigma$ be a developable.

Let

$$
\tau = \lambda y + \mu\sigma
$$

represent the point in which $P_y P_\sigma$ intersects the edge of regression of the developable. Then, since $P_y P_\sigma$ must be tangent to the edge of regression, we shall have

$$
\tau' = \alpha y + \beta\sigma,
$$

or

$$
(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' + \mu\sigma' = 0.
$$

1) cf. Chapter V, equ. (15).

But according to (29)

$$\sigma' = -(P_4 - P_3')y - 3P_3 y',$$

so that

$$(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' - \mu[(P_4 - P_3')y + 3P_3 y'] = 0,$$

where for y' we could also write z. Such a relation between P_y, P_z, P_σ would, however, make these three points collinear, and therefore $P_y, P_z, P_\varrho, P_\sigma$ co-planar, unless all of the coefficients are zero. We have seen, however, that these four points are coplanar only at points P_y whose osculating plane is stationary. Consequently

$$\lambda' - \alpha - \mu(P_4 - P_3') = 0, \quad \mu' - \beta = 0, \quad \lambda - 3\mu P_3 = 0,$$

whence

$$\lambda = 3\mu P_3, \quad \beta = \mu', \quad \alpha = \lambda' - \mu(P_4 - P_3').$$

We see, therefore, that

$$\tau = 3P_3 y + \sigma \tag{30}$$

represents the edge of regression of the developable to which the ruled surface generated by $P_y P_\sigma$ reduces when $P_2 = 0$.

If $p_1 = 0$ and $P_2 = 0$, equations (2) and (10) show that the most general transformations of the variables, which do not disturb these conditions, satisfy the equations

$$\frac{\lambda'}{\lambda} + \frac{3}{2}\frac{\xi''}{\xi'} = 0, \quad \mu = \eta' - \frac{1}{2}\eta^2 = 0,$$

which give on integration

$$\lambda = \frac{C}{(\xi')^{\frac{3}{2}}}, \quad \eta = \frac{-2c}{1+cx}.$$

If we transform τ under this assumption, we find that it is converted into

$$\bar{\tau} = \frac{1}{(\xi')^2\lambda}\left[\sigma + \frac{3}{2}\eta\varrho + \frac{3}{2}\eta^2 z + \left(\frac{3}{4}\eta^3 + 3P_3\right)y\right], \tag{30a}$$

where η may have any numerical value.

Let us recapitulate. *The ruled surfaces generated by $P_y P_{\bar\sigma}$ are infinite in number. Their general expression involves an arbitrary function η. Among these surfaces there exists a single infinity of developables. If $P_2 = 0$, the surface generated by $P_y P_\sigma$ is one of these, and the locus of P_τ is its edge of regression, where*

$$\tau = 3P_3 y + \sigma, \tag{30}$$

P_τ being the point where $P_y P_\sigma$ intersects the edge of regression. If we construct all of the ∞^1 lines $P_y P_{\bar\sigma}$ through P_y, which are generators of the above mentioned family of developables, and mark upon each of them the point $P_{\bar\tau}$ where it intersects the cuspidal edge of the developable

to which it belongs, the locus of these points is a twisted cubic curve. The equations of this curve, referred to a parameter η and to the fundamental tetrahedron $P_y P_z P_\varrho P_\sigma$, are

$$(31) \qquad x_1 = 3P_3 + \frac{3}{4}\eta^3, \quad x_2 = \frac{3}{2}\eta^2, \quad x_3 = \frac{3}{2}\eta, \quad x_4 = 1.$$

We shall see later that this cubic has five consecutive points in common with the curve C_y at P_y, i. e. that it has at this point with C_y a contact of the fourth order. We shall speak of it as the *torsal cubic of P_y*, on account of its connection with the developables which we have just been considering.

Equations (31) give the parametric equations of the torsal cubic referred to a special tetrahedron of reference for which $P_2 = 0$. We shall need its equations in a more general form. These may be easily obtained. Consider the expression

$$(32) \quad \lambda = \left(3\Theta_3 + \frac{3}{10}P_2' + \frac{6}{5}P_2 t + \frac{3}{4}t^3\right)y + \left(\frac{6}{5}P_2 + \frac{3}{2}t^2\right)z + \frac{3}{2}t\varrho + \sigma$$

in which t may, for the moment, be regarded as a parameter independent of x. Denote by $\bar{\lambda}$ the corresponding expression formed from the quantities P_2, P_3, etc., $\bar{y}$, $\bar{z}$, $\bar{\varrho}$, σ after the general transformation $\xi = \xi(x)$. We shall find that $(\xi')^{1}\bar{\lambda}$ is equal to an expression of the form (32) in which, however, t_1 takes the place of t, where $t\xi' + \eta = t_1$. But, of course, this transformation may be chosen so as to make $\bar{P}_2 = 0$, which would make $\bar{\lambda}$ identical with τ except for the notation.

We see, therefore, that the expression λ, or the equations

$$(33) \qquad \begin{aligned} x_1 &= 3\Theta_3 + \frac{3}{10}P_2' + \frac{6}{5}P_2\eta + \frac{3}{4}\eta^3, \\ x_2 &= \frac{6}{5}P_2 + \frac{3}{2}\eta^2, \\ x_3 &= \frac{3}{2}\eta, \quad x_4 = 1 \end{aligned}$$

represent the torsal cubic referred to the fundamental tetrahedron $P_y P_z P_\varrho P_\sigma$ when this is chosen in as general a way as is compatible with its definition.

If, in (32), t is chosen as a function of x, as x varies we obtain a curve on the surface formed by the totality of torsal cubics. If in particular t satisfies, as function of x, the differential equation

$$t' - \frac{1}{2}t^2 = \frac{6}{5}P_2,$$

we obtain the cuspidal edge of one of the developables.

§ 4. The osculating cubic, conic and linear complex.

A space cubic is determined by six of its points provided that no four of these points are coplanar. If, therefore, we take upon C_y, besides P_y, five other points, we shall in general obtain a perfectly definite space cubic determined by these six points. As these points approach coincidence with P_y, the cubic will in general approach a limit, which shall be called the *osculating cubic.* We proceed to find its equations.

Let P_y correspond to the value of $x = a$, which we shall suppose to be an ordinary point for our differential equation. Then y may be developed by Taylor's theorem into a series proceeding according to powers of $x - a$. By putting $x - a = x'$ the development will be in powers of x'. We may, therefore, assume in the first place that $a = 0$. Let us assume further that $p_1 = 0$ and $P_2 = 0$. Then we shall have from (5) and (6),

$$(34)\quad \begin{aligned} &y' = z, \quad y'' = \varrho, \quad y^{(3)} = -P_3 y + \sigma, \quad y^{(4)} = -P_4 y - 4P_3 z, \\ &y^{(5)} = -P_4' y - (P_4 + 4P_3')z - 4P_3\varrho. \end{aligned}$$

In accordance with the definition of our coordinates, we denote the coefficients of y, z, ϱ, σ in this expansion carried as far as x^5, by y_1, y_2, y_3, y_4. We may, of course, multiply these quantities by a common factor, since the coordinates are homogeneous. We shall multiply by 120 so as to clear of fractions. This gives

$$(35)\quad \begin{aligned} y_1 &= 120 - 20P_3x^3 - 5P_4x^4 - P_4'x^5 + \cdots \\ y_2 &= 120x - 20P_3x^4 - (4P_3' + P_4)x^5 + \cdots \\ y_3 &= 60x^2 - 4P_3x^5 + \cdots \\ y_4 &= 20x^3 + \cdots. \end{aligned}$$

We see at once that the following equations are exact up to terms no higher than the fifth order,

$$(36)\quad \begin{aligned} 3y_2y_4 - 2y_3^2 &= 0, \\ 5(2y_1y_3 - y_2^2) - 6P_3y_3y_4 &= 0. \end{aligned}$$

These same equations must be satisfied by the coordinates of any point of the osculating cubic, since this must have contact of the fifth order with C_y at P_y. *They are, therefore, its equations, referred to this special tetrahedron of reference. In terms of a parameter t we may write*

$$(37)\quad x_1 = 15 + 12P_3t^3, \quad x_2 = 30t, \quad x_3 = 30t^2, \quad x_4 = 20t^3.$$

The equation
$$3x_2x_4 - 2x_3^2 = 0$$

is that of a cone, whose vertex is P_y and which contains the osculating cubic. It may also be obtained by determining that cone of the

second order, with its vertex at P_y, which has the closest possible contact with C_y viz. contact of the fifth order. We shall speak of it as the *osculating cone.* We notice at once that the torsal cubic also lies upon the osculating cone. This is shown by equations (31), which are referred to the same system of coordinates as that employed here. If we put in (31) $\eta = \frac{1}{t}$ and if we multiply by $20t^3$, (31) becomes

(31a) $\quad x_1 = 15 + 60P_3t^3, \quad x_2 = 30t, \quad x_3 = 30t^2, \quad x_4 = 20t^3,$

which differs from (37) only in having $5P_3$ in place of P_3.

By a method of reasoning precisely similar to that of the last paragraph, we find that *the expression*

$$(38) \quad (12P_3 - 12P_2' + 24P_2\tau + 15\tau^3)y + 20\left(\frac{6}{5}P_2 + \frac{3}{2}\tau^2\right)z + 30\tau\varrho + 20\sigma$$

represents an arbitrary point of the osculating cubic, when the tetrahedron of reference is not restricted to the condition $P_2 = 0$. For, this expression remains invariant under the general transformation $\xi = \xi(x)$, and reduces to (37) for $P_2 = 0$, if $\tau = \frac{1}{t}$.

The equation of the plane, which osculates the osculating cubic at the point whose parameter is τ, turns out to be

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

where

$$(39) \quad \begin{aligned} u_1 &= -20, \quad u_2 = 30\tau, \quad u_3 = 16P_2 - 30\tau^2, \\ u_4 &= 12P_3 - 12P_2' - 36P_2\tau + 15\tau^3. \end{aligned}$$

For every value of τ, this intersects the osculating plane, $x_4 = 0$, in a straight line

$$-20x_1 + 30\tau x_2 + (16P_2 - 30\tau^2)x_3 = 0.$$

The envelope of these lines will be obtained by eliminating τ between this equation and that obtained from it by partial differentiation with respect to τ; the latter equation is

$$30x_2 - 60\tau x_3 = 0.$$

We thus find

$$(40) \quad -40x_1x_3 + 15x_2^2 + 32P_2x_3^2 = 0,$$

the equation of the osculating conic, which may be defined as a part of the intersection of the developable of the osculating cubic with the osculating plane. The other part of this intersection is the tangent, which must be counted twice.

It is not without interest to verify that (37) represents the osculating cubic by another method. We find, from (35), that the nonhomogeneous coordinates of the points of C_y in the vicinity of P_y are

$$
(41)\qquad
\begin{aligned}
&\frac{y_2}{y_1} = x - \frac{1}{30}(P_3' - P_4)x^5 + \cdots,\\
&\frac{y_3}{y_1} = \frac{1}{2}x^2 + \frac{1}{20}P_3x^5 + \cdots, \quad \frac{y_4}{y_1} = \frac{1}{6}x^3 + \cdots.
\end{aligned}
$$

From (37), we find for the points of the osculating cubic

$$\frac{x_2}{x_1} = 2t - \frac{8}{5}P_3t^4 + \cdots, \quad \frac{x_3}{x_1} = 2t^2 - \frac{8}{5}P_3t^5 + \cdots, \quad \frac{x_4}{x_1} = \frac{4}{3}t^3 + \cdots.$$

If we put

$$t = \frac{1}{2}x\left[1 + \frac{1}{10}P_3x^3 - \frac{1}{30}(P_3' + P_4)x^4 + \cdots\right],$$

the two expansions coincide up to terms of the fifth order. For the torsal cubic we have, according to (31a),

$$\frac{x_2}{x_1} = 2t - 8P_3t^4 + \cdots, \quad \frac{x_3}{x_1} = 2t^2 - 8P_3t^5 + \cdots, \quad \frac{x_4}{x_1} = \frac{4}{3}t^3 + \cdots.$$

If we put into these equations

$$t = \frac{1}{2}x(1 + ax + bx^2 + cx^3 + dx^4 + \cdots),$$

we find that these expansions will agree with (41) up to terms of the fourth order if

$$a = 0, \quad b = 0, \quad c = \frac{1}{2}P_3,$$

but that it is impossible to make them agree with (41) any further unless $P_3 = 0$. *In general, therefore, the torsal cubic has with C_y a contact of the fourth order. Only if $\Theta_3 = 0$ may the order of contact be higher. In that case the torsal and osculating cubics coincide.*

We proceed to deduce the equation of the *osculating linear complex*, i. e. of that linear complex determined by five consecutive tangents of the curve. We assume again $p_1 = 0$ and $P_2 = 0$. Denote by Y and Z the expansions of y and z in the vicinity of P_y. Then we have up to terms of the fourth order

$$Y = y\left(1 - \frac{1}{6}P_3x^3 - \frac{1}{24}P_4x^4\right) + z\left(x - \frac{1}{6}P_3x^4\right) + \frac{1}{2}\varrho x^2 + \frac{1}{6}\sigma x^3,$$

$$
\begin{aligned}
Z = y&\left(\quad - \frac{1}{2}P_3x^2 - \frac{1}{6}P_4x^3 - \frac{1}{24}P_4'x^4\right)\\
&+ z\left[1 - \frac{2}{3}P_3x^3 - \frac{1}{24}(4P_3' + P_4)x^4\right]\\
&+ \varrho\left(x - \frac{1}{6}P_3x^4\right) + \sigma\frac{1}{2}x^2.
\end{aligned}
$$

If we denote the coefficients of y, z, ϱ, σ in these two expressions by $y_1, \ldots y_4$ and $z_1, \ldots z_4$ respectively, the Plückerian line-coordinates of the tangent will be

$$\omega_{ik} = y_i z_k - y_k z_i,$$

whence

$$\omega_{12} = 1 - \frac{1}{2} P_3 x^3 + \frac{1}{8} P_4 x^4 + \cdots,$$

$$\omega_{13} = x - \frac{1}{12} P_3 x^4 + \cdots,$$

$$\omega_{14} = \frac{1}{2} x^2 + \cdots, \quad \omega_{23} = \frac{1}{2} x^2 + \cdots,$$

$$\omega_{24} = \frac{1}{3} x^3 + \cdots, \quad \omega_{34} = \frac{1}{12} x^4 + \cdots.$$

Therefore, *the equation of the osculating linear complex, referred to the special tetrahedron of reference, is*

$$\omega_{14} - \omega_{23} = 0. \tag{42}$$

We might have obtained this complex in another way. For, it is clear that the null-system of the osculating cubic will be the same as that determined by the osculating linear complex. We shall, instead, set up the null-system of the torsal cubic in its general form. We shall see that *the linear complex determined by the torsal cubic coincides with the osculating linear complex.*

We have the equations of the torsal cubic

$$\begin{aligned} x_1 &= 60\Theta_3 + 6P_2' + 24P_2\eta + 15\eta^3, \\ x_2 &= 24P_2 + 30\eta^2, \quad x_3 = 30\eta, \quad x_4 = 20. \end{aligned} \tag{43}$$

The coordinates of the plane, which osculates the torsal cubic at the point whose parameter is η, are

$$\begin{aligned} u_1 &= -180, \quad u_2 = 270\eta, \quad u_3 = 144P_2 - 270\eta^2, \\ u_4 &= 540\Theta_3 + 54P_2' - 324P_2\eta + 135\eta^3. \end{aligned} \tag{44}$$

If we put in (43) $\eta = \eta_k$, $(k = 1, 2, 3)$, we obtain three points on the cubic. The coordinates of their plane must be proportional to

$$\begin{aligned} v_1 &= -180, \quad v_2 = 90(\eta_1 + \eta_2 + \eta_3), \\ v_3 &= 144P_2 - 90(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ v_4 &= 540\Theta_3 + 54P_2' - 108P_2(\eta_1 + \eta_2 + \eta_3) + 135\eta_1\eta_2\eta_3, \end{aligned} \tag{45}$$

for, each of these expressions must be a symmetric function of η_1, η_2, η_3 of not higher than the third order, and for $\eta_1 = \eta_2 = \eta_3 = \eta$ we must have v_k proportional to u_k. Similarly, the point in which the three osculating planes at η_1, η_2, η_3 intersect, must have its coordinates proportional to

$$
(46)\qquad
\begin{aligned}
\omega x_1 &= 60\Theta_3 + 6P_2' + 8P_2(\eta_1 + \eta_2 + \eta_3) + 15\eta_1\eta_2\eta_3,\\
\omega x_2 &= 24P_2 + 10(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2),\\
\omega x_3 &= 10(\eta_1 + \eta_2 + \eta_3), \quad \omega x_4 = 20.
\end{aligned}
$$

If we eliminate η_1, η_2, η_3 between (45) and (46), and change slightly the factor of proportionality, we find

$$
(47\text{a})\qquad
\begin{aligned}
\omega v_1 &= -x_4, \quad \omega v_3 = -x_2 + 2P_2x_4,\\
\omega v_2 &= +x_3, \quad \omega v_4 = +x_1 - 2P_2x_3,
\end{aligned}
$$

or

$$
(47\text{b})\qquad
\begin{aligned}
\omega' x_1 &= +2P_2v_2 + v_4, \quad \omega' x_3 = +v_2,\\
\omega' x_2 &= -2P_2v_1 - v_3, \quad \omega' x_4 = -v_1,
\end{aligned}
$$

as the equations of the null-system defined by the torsal cubic.

A point y_1, y_2, y_3, y_4 lies in the plane corresponding to x_1, x_2, x_3, x_4 if

$$\sum_{k=1}^{4} v_k y_k = 0.$$

Therefore, the lines which pass through the point x_1, x_2, x_3, x_4 and lie in the plane corresponding to it in the null-system, satisfy the equation

$$
(48)\qquad \omega_{14} - 2P_2\omega_{34} - \omega_{23} = 0.
$$

If the tetrahedron of reference be so chosen as to make $P_2 = 0$, this equation is identical with (42). *Therefore, (48) represents the osculating linear complex when the tetrahedron of reference is general. The osculating and torsal cubics are curves of this complex.*

If P_2 is finite, the complex (48) is not special We see, therefore, that only those values of x, for which $P_2 = \infty$, can give points of the curve at which five consecutive tangents have a straight line intersector.

Let us proceed to deduce the equation of the osculating linear complex, belonging to a point of C_y infinitesimally close to P_y. If we change x by an infinitesimal amount δx, we find for the coordinates of the vertices of the new tetrahedron of reference

$$
\begin{aligned}
\bar{y} &= y + y'\delta x = (1 - p_1\delta x)y + z\delta x,\\
\bar{z} &= z + z'\delta x = -P_2\delta x\cdot y + (1 - p_1\delta x)z + \varrho\delta x,\\
\bar{\varrho} &= \varrho + \varrho'\delta x = -(P_3 - P_2')\delta x\cdot y - 2P_2\delta x\cdot z + (1 - p_1\delta x)\varrho + \sigma\delta x,\\
\bar{\sigma} &= \sigma + \sigma'\delta x = -(P_4 - P_3')\delta x\cdot y - 3(P_3 - P_2')\delta x\cdot z - 3P_2\delta x\cdot\varrho\\
&\qquad + (1 - p_1\delta x)\sigma.
\end{aligned}
$$

Therefore, if a point has the coordinates $\bar{x}_1, \dots \bar{x}_4$ in the new system of coordinates, and $x_1 \dots x_4$ in the old, we shall have

$$x_1 = (1 - p_1\delta x)\bar{x}_1 - P_2\delta x \cdot \bar{x}_2 - (P_3 - P_2')\delta x \cdot \bar{x}_3 - (P_4 - P_3')\delta x \cdot \bar{x}_4,$$

. . . .

Therefore, the infinitesimal changes in the coordinates, in the sense new minus old, will be

$$\begin{aligned} \delta x_1 &= [p_1 x_1 + P_2 x_2 + (P_3 - P_2')x_3 + (P_4 - P_3')x_4]\delta x, \\ \delta x_2 &= [-x_1 + p_1 x_2 + 2P_2 x_3 + 3(P_3 - P_2')x_4]\delta x, \\ \delta x_3 &= (-x_2 + p_1 x_3 + 3P_2 x_4)\delta x, \\ \delta x_4 &= (-x_3 + p_1 x_4)\delta x. \end{aligned} \tag{49}$$

Referred to the new tetrahedron of reference, the equation of the complex, osculating C_y at the point corresponding to $x + \delta x$, will be

$$\bar{\omega}_{14} - 2\bar{P}_2\omega_{34} - \bar{\omega}_{23} = 0, \tag{50}$$

where

$$\bar{P}_2 = P_2 + P_2'\delta x, \quad \bar{\omega}_{ik} = \bar{x}_i\bar{y}_k - \bar{x}_k\bar{y}_i,$$
$$\bar{x}_i = x_i + \delta x_i, \qquad \bar{y}_i = y_i + \delta y_i,$$

if $\bar{x}_i$ and $\bar{y}_i$ denote the coordinates of two points on a line of the complex referred to the new tetrahedron of reference. Making the calculations, we find

$$\begin{aligned} \bar{\omega}_{14} &= \omega_{14} + [-\omega_{13} + 2p_1\omega_{14} - P_2\omega_{42} + (P_3 - P_2')\omega_{34}]\delta x, \\ \bar{\omega}_{34} &= \omega_{34} + (2p_1\omega_{34} + \omega_{42})\delta x, \\ \omega_{23} &= \omega_{23} + [2p_1\omega_{23} - 3P_2\omega_{12} \quad \omega_{13} - 3(P_3 - P_2')\omega_{34}]\delta x. \end{aligned}$$

If we substitute in (50), we find as the *equation of the linear complex osculating C_y at a point infinitesimally close to P_y*,

$$(\omega_{14} - 2P_2\omega_{34} - \omega_{23})(1 + 2p_1\delta x) + 4\Theta_3\omega_{34}\delta x = 0. \tag{51}$$

This coincides with the linear complex osculating C_y at P_y, if and only if $\Theta_3 = 0$.

Therefore, *if the invariant Θ_3 vanishes identically, the tangents of the curve C_y belong to a linear complex.* If it does not vanish identically, those values of x, for which it does vanish, correspond to points of the curve at which the osculating linear complex hyperosculates the curve.

This result may also be obtained by setting up the linear differential equation of the sixth order satisfied by the six line coordinates of the tangent

$$\omega_{ik} = y_i y_k' - y_k y_i',$$

and noting that this reduces to the fifth order if and only if $\Theta_3 = 0$. This is the method of *Halphen*.[1])

1) Acta Mathematica, vol. 3 (1883).

A former result may now be stated as follows. *The osculating and torsal cubics of all points of a curve coincide, if and only if the curve belongs to a linear complex.*

§ 5. Geometrical definition of the fundamental tetrahedron of reference.

We have seen that there exists for every point of the curve C_y a tetrahedron whose vertices $P_y, P_z, P_\varrho, P_\sigma$ are determined by the choice of the independent variable x. In order that we may be able to obtain a clear insight into the geometry of the curve, it is necessary that we may be able to define this tetrahedron by purely geometrical considerations. As a consequence of our preceding results we are now able to do this.

We have already noticed that P_z is a point on the tangent, and that by a properly chosen transformation $\xi = \xi(x)$ it may be transformed into any other point of the tangent. When the independent variable has been definitively chosen, we obtain, therefore, a point P_z on the tangent which is not, in general, distinguished by any geometrical property from any other point of the tangent. Its position may serve as a geometrical image of the independent variable.

Consider the osculating conic.

$$x_4 = 0, \quad 40x_1x_3 - 32P_2x_3^2 - 15x_2^2 = 0.$$

The polar of any point $(x_1', x_2', x_3', 0)$ of the osculating plane with respect to it, is the straight line

$$x_4 = 0, \quad 20x_3'x_1 - 15x_2'x_2 + (20x_1' - 32P_2x_3')x_3 = 0.$$

Therefore, the polar of P_z, whose coordinates are $(0, 1, 0, 0)$, is the line $x_2 = 0, \quad x_4 = 0$. In other words:

The line P_yP_ϱ is the polar of P_z with respect to the osculating conic.

We shall speak of the curves C_z, C_ϱ, C_σ as the *derivative curves of C_y with respect to x, of the first, second and third kind* respectively. The ruled surfaces, which are obtained by joining the points of C_y to the corresponding points of C_z, C_ϱ, C_σ, shall be called *derivative ruled surfaces of the first, second and third kind* respectively. Then, the derivative ruled surface of the first kind is unique. It is simply the developable whose cuspidal edge is C_y. Let us consider the derivative ruled surface S of the second kind generated by P_yP_ϱ. The curve C_y is, of course, an asymptotic curve upon it. This surface is characterized by the equations (20), where p_1 has been assumed equal to zero.

According to the general theory of ruled surfaces[1]), the asymptotic tangents to S at the points P_y and P_ϱ are obtained by joining these points to
$$2z \quad \text{and} \quad 2\sigma - 4P_2 z + 2P_3 y$$
respectively. Therefore, the asymptotic tangent to S at any point $(\alpha_1, 0, \alpha_3, 0)$ of $P_y P_\varrho$ joins this point to
$$2P_3\alpha_3 y + (2\alpha_1 - 4P_2\alpha_3)z + 2\alpha_3\sigma.$$
Hence, the equation of the plane tangent to S at $(\alpha_1, 0, \alpha_3, 0)$ is
$$-\alpha_3 x_2 + (\alpha_1 - 2P_2\alpha_3)x_4 = 0.$$
To the same point of $P_y P_\varrho$ there corresponds a plane in the osculating linear complex According to (47a), this is the plane
$$\alpha_3 x_2 + (\alpha_1 - 2P_2\alpha_3)x_4 = 0.$$
Therefore, if at any point of the generator of the derived ruled surface of the second kind we construct the tangent plane as well as the plane which corresponds to it in the osculating linear complex, these planes form an involution. The double planes of this involution are the osculating plane $(x_4 = 0)$, and a plane $(x_2 = 0)$ which contains P_σ, the point of the derivative curve of the third kind which corresponds to P_y.

The point which corresponds to this latter plane, is

(52) $$\beta = 2P_2 y + \varrho.$$

According to (47a) we have further, corresponding to the point P_z or $(0, 1, 0, 0)$, the plane $x_3 = 0$, which also contains P_σ. The line $P_y P_\sigma$ is now completely determined, as follows.

The generator of the derived ruled surface of the third kind is the intersection of the following two planes; 1st the plane corresponding to P_z in the osculating linear complex; 2d that plane which is tangent to the derived ruled surface of the second kind at the same point which corresponds to it in the osculating linear complex.

It still remains to determine the position of P_ϱ and P_σ on the lines $P_y P_\varrho$ and $P_y P_\sigma$.

The osculating conic intersects $P_y P_\varrho$ in P_y and in P_α where

(53) $$\alpha = 4P_2 y + 5\varrho.$$

The cross-ratio of the four points $P_y, P_\varrho, P_\alpha, P_\beta$, is
$$(\alpha, y, \beta, \varrho) = \frac{5}{2}.$$

If, upon the generator of the derived ruled surface of the second kind, there be marked its intersections with the osculating conic, and the

1) Cf. Chapter VI.

point P_β whose tangent plane coincides with the plane corresponding to it in the osculating linear complex, the point P_ϱ is determined by the condition that the cross-ratio of these four points shall be equal to $\frac{5}{2}$.

If $P_2 = 0$ this definition of P_ϱ breaks down. In that case, however, P_α and P_β coincide with P_ϱ. Therefore, *if the derived ruled surface of the third kind is a developable, P_ϱ is that point on the generator of the derived surface of the second kind where this generator intersects the osculating conic for the second time. At this point the plane, tangent to the ruled surface, and the plane, corresponding to it in the osculating linear complex coincide.*

If we use the notations of the theory of ruled surfaces[1]), we find from (20),

$$u_{12} = 4, \quad u_{21} = 8P_3' - 4P_4 + 8P_2^2, \; u_{11} - u_{22} = 16P_2,$$

$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = -64(P_4 - 2P_3' - 6P_2^2).$$

But $u_{11} - u_{22} = 0$ is the condition that C_y and C_ϱ shall be harmonically divided by the branches of the flecnode curve of the ruled surface, while $(u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0$ is the condition under which the two branches of the flecnode curve coincide.[2]) Therefore, we obtain the following theorem.

If the derived ruled surface of the third kind is a developable, the intersections of the generator of the derived ruled surface of the second kind with the osculating conic give rise to two curves upon this surface harmonically conjugate with respect to the two branches of its flecnode curve.

If $\Theta_4 = 0$, the second intersection of the generator of this surface with the osculating conic is a point of its flecnode curve. Moreover, the two branches of the flecnode curve must then coincide.

It is to be noted that we have here a *geometrical interpretation for the invariant equation $\Theta_4 = 0$.* We shall find two other, quite different, interpretations for this condition later on.

We may, if we wish, make use of the torsal cubic in our further constructions. For, it is now defined entirely by geometrical considerations. If, in fact, we trace upon the developable, whose edge of regression is C_y, an arbitrary curve C_z, we now know how the corresponding ruled surfaces of the second and third kind may be constructed They depend upon an arbitrary function of x, as does the curve C_z. Among the surfaces of the third kind there exists a single one-parameter family of developables. Upon that generator of each of these developables which passes through P_y, we mark the

1) Chapter IV, equ (20).

2) Cf. Chapter VI.

point where it intersects the cuspidal edge of the developable to which it belongs. The locus of these points is the torsal cubic.

We notice incidentally that *the reduction of equation* (1) *to the Laguerre-Forsyth canonical form is equivalent to the determination of one of the developables of the third kind.* Since this reduction is made by solving an equation of the *Riccati* form, we notice further the following theorem. *The four curves on the developable of C_y, which correspond to any four of the developables of the third kind, intersect all of the tangents of C_y in point-rows of the same cross-ratio.*

Let us consider the developable surface of the torsal cubic, which is given by equations (44). We are going to find its intersection with the plane $P_y P_z P_\sigma$, or $x_3 = 0$. The intersection of the plane $u_1, \dots u_4$, which osculates the cubic at the point whose parameter is η, with the plane $x_3 = 0$ is the line

$$-180x_1 + 270\eta x_2 + (540\Theta_3 + 54P_2' - 324P_2\eta + 135\eta^3)x_4 = 0$$

of this plane. As η changes, this line envelops a curve, the required intersection. Its equation will be found by eliminating η between the above equation and this other one

$$270x_2 + (-324P_2 + 405\eta^2)x_4 = 0$$

obtained from it by differentiation with respect to η. This elimination may be easily performed. The result is

$$F = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 \\ + 15x_4\{10x_1 - (30\Theta_3 + 3P_2')x_4\}^2 = 0. \tag{54}$$

This plane cubic together with the tangent $P_y P_z$ gives the complete intersection of the plane $P_y P_z P_\sigma$ with the developable of the torsal cubic. It has a cusp at P_z, and the equation of its cusp tangent is

$$5x_1 - 6x_4 = 0,$$

as one may find by the general theory of plane curves. The cusp tangent intersects $P_y P_\sigma$ in the point

$$\varkappa = 24y + 20\sigma. \tag{55}$$

The tangent to the plane cubic at P_y is

$$10x_2 + 3x_4 = 0.$$

It intersects the cubic again in the point

$$(12 + 30\Theta_3 + 3P_2')y - 6z + 20\sigma.$$

If this point be joined to P_z by a straight line, the latter will intersect $P_y P_\sigma$ in the point

$$\lambda = (12 + 30\Theta_3 + 3P_2')y + 20\sigma. \tag{56}$$

The plane $P_y P_z P_\sigma$ is tangent to the torsal cubic. It intersects it once more in the point corresponding to $\eta = 0$, viz.:

$$(60\Theta_3 + 6P_2')y + 24P_2 z + 20\sigma.$$

A line joining this point to P_z intersects $P_y P_\sigma$ in

$$\mu = (60\Theta_3 + 6P_2')y + 20\sigma. \tag{57}$$

Consider the four points $P_\varkappa$, P_λ, P_μ and P_y. We have

$$\lambda = \frac{\mu + \varkappa}{2},$$

$$(30\Theta_3 + 3P_2' - 12)y = \frac{\mu - \varkappa}{2},$$

so that P_λ is the harmonic conjugate of P_y with respect to $P_\varkappa$ and P_μ.

The osculating cubic differs from the torsal cubic only in having $2(P_3 - P_2')$ in place of $10\Theta_3 + P_2'$. Consequently, the plane cubic in which its developable intersects the plane $P_y P_z P_\sigma$ is

$$\begin{aligned} F = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 \\ + 15x_4\{10x_1 - 6(P_3 - P_2')x_4\}^2 = 0. \end{aligned} \tag{58}$$

If we denote by $P_{\bar{\varkappa}}$, $P_{\bar{\lambda}}$, $P_{\bar{\mu}}$ the points constructed with respect to this curve in the same way as $P_\varkappa$, P_λ and P_μ were with respect to $F = 0$, we find

$$\begin{aligned} \bar{\varkappa} = \varkappa, \quad \bar{\lambda} &= (12 + 6P_3 - 6P_2')y + 20\sigma, \\ \bar{\mu} &= (12P_3 - 12P_2')y + 20\sigma, \end{aligned} \tag{59}$$

the cusp and its tangent being common to the two curves, as well as the tangent at P_y. I refrain from formulating explicitly the various theorems which may be obtained from these equations.

In order to obtain a simple construction for P_σ, we shall consider, finally, the developable generated by the motion of the plane $P_y P_\varrho P_\sigma$. The equation of this plane is $x_2 = 0$. As x changes into $x + \delta x$, y, z, ϱ change into $y + y'\delta x$, $\varrho + \varrho'\delta x$, $\sigma + \sigma'\delta x$ respectively, where y', ϱ', σ' are given by equations (5). The equation of the plane of these points, referred to the tetrahedron $P_y, P_z, P_\varrho, P_\sigma$, is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 - p_1\delta x & \delta x & 0 & 0 \\ -(P_3 - P_2')\delta x & -2P_2\delta x & 1 - p_1\delta x & \delta x \\ -(P_4 - P_3')\delta x & -3(P_3 - P_2')\delta x & -3P_2\delta x & 1 - p_1\delta x \end{vmatrix} = 0,$$

which becomes, when developed,

$$x_1\delta x - x_2(1 - 3p_1\delta x) - 2x_3P_2\delta x - 3x_4(P_3 - P_2')\delta x = 0.$$

Therefore, the equations of the generator of the developable, generated by the motion of the plane $P_y P_\varrho P_\sigma$, are

$$x_2 = 0, \quad x_1 - 2P_2 x_3 - 3(P_3 - P_2')x_4 = 0. \tag{60}$$

It intersects the generator $P_y P_\varrho (x_2 = x_4 = 0)$ of the derived ruled surface of the second kind in that point

$$\beta = 2P_2 y + \varrho, \tag{52}$$

whose tangent plane coincides with the plane corresponding to it in the osculating linear complex. Its intersection with $P_y P_\sigma$, the generator of the derived ruled surface of the third kind, is

$$\gamma = 3(P_3 - P_2')y + \sigma. \tag{61}$$

The generator of the developable joins P_β to P_γ. We wish to determine its edge of regression. If

$$\delta = l\beta + m\gamma$$

is the point where $P_\beta P_\gamma$ meets the edge of regression, we must have

$$\delta' = r\beta + s\gamma, \quad \text{or} \quad l\beta' + m\gamma' = \bar{r}\beta + \bar{s}\gamma.$$

We proceed to determine the ratio of l to m. We find

$$\beta' = (3P_2' - 2p_1 P_2 - P_3)y - p_1\varrho + \sigma,$$
$$\gamma' = -[P_4 - P_3' + 3P_2'' + 3p_1(P_3 - P_2')]y - 3P_2\varrho - p_1\sigma.$$

We may eliminate ϱ and σ by (52) and (61). This gives

$$\beta' = -4\Theta_3 y - p_1\beta + \gamma,$$
$$\gamma' = -(P_4 - 4P_3' + 3P_2'' - 6P_2^2)y - 3P_2\beta - p_1\gamma.$$

We may, therefore, put

$$l = P_4 - 4P_3' + 3P_2'' - 6P_2^2,$$
$$m = -4\Theta_3,$$

so that

$$\delta = (P_4 - 4P_3' + 3P_2'' - 6P_2^2)\beta - 4\Theta_3\gamma \tag{62}$$

gives the edge of regression. This gives the following theorem. *The developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, has its edge of regression upon the derived ruled surface of the second kind, if and only if the curve C_y belongs to a linear complex.*

We may write, in place of (61),

$$\gamma = 60(P_3 - P_2')y + 20\sigma.$$

We have from (59)

$$\bar{\mu} = 12(P_3 - P_2')y + 20\sigma.$$

Therefore, the cross-ratio of the four points P_γ, P_y, $P_{\bar{\mu}}$ and $P_{\bar{\sigma}}$ is

$$(\gamma, y, \bar{\mu}, \sigma) = 5.$$

We have found finally a geometrical definition for P_σ, which we may recapitulate as follows. *The plane of the tangent and the generator of the third derived ruled surface intersects the osculating cubic in P_y counted twice and one other point. If the latter point be joined to P_z by a straight line, we obtain a certain point $P_{\bar{\mu}}$ as the intersection of this line with $P_y P_\sigma$. The generator of the developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, intersects $P_y P_\sigma$ in another point P_γ. P_σ may now be found as that point of $P_y P_\sigma$ which makes the cross-ratio*

$$(P_\gamma, P_y, P_{\bar{\mu}}, P_\sigma) = 5.$$

We have shown how to construct the fundamental tetrahedron when P_z is given. If P_ϱ is given, P_z can be found at once as the pole of $P_y P_\varrho$ with respect to the osculating conic. If P_σ is given, we may find first its polar plane with respect to the osculating linear complex, which is

$$-x_1 + 2P_2 x_3 = 0,$$

and therefore passes through P_z, but not through P_y. P_z can therefore be found at once as the intersection of this plane with the tangent to C_y at P_y.

We see, therefore, that *any one of the three points P_z, P_ϱ, P_σ determines uniquely the others.*

§ 6. Some further properties of the derived ruled surfaces of the second and third kind.

Let us suppose $p_1 = P_2 = 0$, so that the derived ruled surface of the third kind is a developable, and let us consider the derived ruled surface of the second kind which corresponds to it. We proceed to deduce the equation of its osculating linear complex.

Let Y and R denote the developments of y and ϱ in the vicinity of the ordinary point $x = a$, and replace again $x - a$ by x in the developments. Then we shall find

$$Y = y_1 y + y_2 z + y_3 \varrho + y_4 \sigma,$$

$$R = \varrho_1 y + \varrho_2 z + \varrho_3 \varrho + \varrho_4 \sigma,$$

where

$$
(63) \quad
\begin{aligned}
\varrho_1 &= -P_3 x - \tfrac{1}{2} P_4 x^2 - \tfrac{1}{6} P_4' x^3 + \tfrac{1}{24}(4P_3^2 - P_4'')x^4 + \cdots,\\
\varrho_2 &= -2P_3 x^2 - \tfrac{1}{6}(4P_3' + P_4)x^3 - \tfrac{1}{12}(2P_3'' + P_4')x^4 + \cdots,\\
\varrho_3 &= 1 - \tfrac{2}{3} P_3 x^3 - \tfrac{1}{24}(8P_3' + P_4)x^4 + \cdots,\\
\varrho_4 &= x - \tfrac{1}{6} P_3 x^4 + \cdots,
\end{aligned}
$$

while $y_1, \ldots y_4$ have been computed before. Denote by

$$\omega_{ik} = y_i \varrho_k - y_k \varrho_i$$

the Plückerian coordinates of the line joining the two points.

We find

$$
\begin{aligned}
\omega_{12} &= -P_3 x^2 - \tfrac{1}{3}(2P_3' - P_4)x^3 - \tfrac{1}{12}(2P_3'' - P_4')x^4 + \cdots,\\
\omega_{13} &= 1 - \tfrac{1}{3} P_3 x^3 - \tfrac{1}{6}(2P_3' - P_4)x^4 + \cdots,\\
\omega_{14} &= x - \tfrac{1}{6} P_3 x^4 + \cdots,\\
\omega_{23} &= x + \tfrac{1}{6} P_3 x^4 + \cdots,\\
\omega_{24} &= x^2 + \cdots, \qquad \omega_{34} = \tfrac{1}{3} x^3 + \cdots.
\end{aligned}
$$

Let

$$a\omega_{12} + b\omega_{13} + c\omega_{14} + d\omega_{23} + e\omega_{24} + f\omega_{34} = 0$$

be the equation of the osculating linear complex of the surface in question. Then, the coefficients of all powers of x up to and including x^4 must be zero, if we substitute the above developments for ω_{ik} into the left member. This gives us the following equations:

$$
\begin{aligned}
& b = 0, \quad c + d = 0, \quad -P_3 a + e = 0,\\
& -\tfrac{1}{3}(2P_3' - P_4)a - \tfrac{1}{3} P_3 b + \tfrac{1}{3} f = 0,\\
& -\tfrac{1}{12}(2P_3'' - P_4')a - \tfrac{1}{6}(2P_3' - P_4)b - \tfrac{1}{6} P_3 c + \tfrac{1}{6} P_3 d = 0,
\end{aligned}
$$

whence the ratios of the coefficients may be easily deduced.

We find thus the equation of the linear complex osculating the derived ruled surface of the second kind which corresponds to a developable of the third kind; it is

$$
(64) \quad
\begin{aligned}
-4P_3\omega_{12} - (P_4' - 2P_3'')(\omega_{14} - \omega_{23}) - 4P_3^2\omega_{24}&\\
+ 4(P_4 - 2P_3')P_3\omega_{34} = 0.&
\end{aligned}
$$

It coincides with the osculating linear complex of C_y, if and only if $P_3 = 0$, i. e. if C_y belongs to a linear complex. This result is also obvious for geometrical reasons.

The coordinates v_k of the plane, which corresponds to a point x_1, x_2, x_3, x_4, in the linear complex (64), are given by

$$
(65)\qquad
\begin{aligned}
\omega v_1 &= \quad * \quad + 4P_3 x_2 + \quad * \quad + (P_4' - 2P_3'')x_4,\\
\omega v_2 &= -4P_3 x_1 + \quad * \quad - (P_4' - 2P_3'')x_3 + 4P_3^2 x_4,\\
\omega v_3 &= \quad * \quad + (P_4' - 2P_3'')x_2 + \quad * \quad - 4P_3(P_4 - 2P_3')x_4,\\
\omega v_4 &= -(P_4' - 2P_3'')x_1 - 4P_3^2 x_2 + 4P_3(P_4 - 2P_3')x_3 + \quad *,
\end{aligned}
$$

where ω is a proportionality factor.

Let us consider at the same time the osculating linear complex of C_y. The lines common to the two complexes form a congruence whose directrices we propose to find. This we can do quite easily by writing down the equations which express that, for a point on one of the directrices, the two planes corresponding to it in the two complexes must coincide. The right members of (65) must, for such a point, be equal to

$$-\omega x_4, \quad +\omega x_3, \quad -\omega x_2, \quad +\omega x_1$$

respectively, where ω is a proportionality factor.

The four equations obtained in this way can be satisfied only if their determinant vanishes, which gives

$$(66)\qquad 2P_3'' - P_4' - \omega = \pm 4P_3\sqrt{2P_3' - P_4},$$

whence the following equations for the two directrices

$$
\begin{aligned}
&\pm\sqrt{2P_3' - P_4}\,x_1 - P_3 x_2 - (2P_3' - P_4)x_3 = 0,\\
&-x_2 \pm \sqrt{2P_3' - P_4}\,x_4 = 0,\\
&-x_1 \pm \sqrt{2P_3' - P_4}\,x_3 + P_3 x_4 = 0,
\end{aligned}
$$

of which three equations only two are independent, and where we have assumed $P_3 \neq 0$. In fact, if P_3 were zero the two complexes would coincide and the congruence would be indeterminate.

Since we have assumed $P_2 = 0$, the quantity under the square root is $-\Theta_4$. We find a second interpretation for the condition $\Theta_4 = 0$. *If $\Theta_4 = 0$, the congruence has coincident directrices.* We may combine this with our former result to the following theorem.

Choose as derived ruled surface of the third kind one of the developables of the single family which exists. Consider the osculating linear complex of the corresponding ruled surface S of the second kind. Let the directrices of the congruence, which this complex has in common with the osculating linear complex of the fundamental curve C_y, coincide. Then the two branches of the flecnode curve of S coincide, and the generator of S which passes through P_y will intersect the osculating conic of C

in P_y and a second point, whose locus is the flecnode curve of the surface S.

I refrain from formulating the converse. The above conditions are fulfilled if and only if $\Theta_4 = 0$.

Let us consider one of the directrices (67), for example that one which corresponds to the plus sign of the square root. Then we see that

$$(68) \qquad \begin{aligned} \alpha &= P_3\sqrt{2P_3' - P_4}\, y + P_3 \varrho, \\ \beta &= \sqrt{2P_3' - P_4}\,(P_3 y + \sqrt{2P_3' - P_4}\, z + \sigma) \end{aligned}$$

are two points on the directrix. We have multiplied each expression by a factor so as to have α and β of the same weight. If now we change the independent variable, but in such a way as not to disturb the condition $P_2 = 0$, we shall get in (64) a single infinity of complexes, and in (67) two families of lines, the directrices of the single infinity of congruences which thus result. We are going to study, to some extent, the two ruled surfaces thus generated.

Put

$$(69) \qquad k = \sqrt{2P_3'' - P_4}.$$

Making the transformations which preserve $P_2 = 0$, for which we must have

$$\mu = \eta' - \frac{1}{2}\eta^2 = 0,$$

we find that α and β are transformed into $\bar{\alpha}$ and $\bar{\beta}$, where

$$\begin{aligned} (\xi')^5 \bar{\alpha} &= \alpha + 2\eta P_3 z + \frac{3}{2}\eta^2 P_3 y, \\ (\xi')^5 \bar{\beta} &= \beta + \frac{3}{2}\frac{k}{P_3}\eta\alpha + k\left(\frac{3}{2}\eta^2 z + \frac{3}{4}\eta^3 y\right). \end{aligned}$$

The point $m\alpha + n\bar{\beta}$ will be an arbitrary point on the line joining P_α and $P_{\bar{\beta}}$. We find, therefore, the equations of our surface, referred to two parameters η and $\frac{m}{n}$,

$$(70) \qquad \begin{aligned} x_1 &= \frac{3}{2} m P_3 \eta^2 + (m + n) P_3 k + n\left(\frac{3}{2}\eta k^2 + \frac{3}{4}\eta^3 k\right), \\ x_2 &= 2m\eta P_3 + n\left(k^2 + \frac{3}{2}\eta^2 k\right), \\ x_3 &= m P_3 + \frac{3}{2} k n \eta, \\ x_4 &= n k. \end{aligned}$$

The nature of this surface may be easily determined. Returning to the two curves C_α and C_β upon it, we find the equations for C_α to be

$$\alpha_1 = k + \frac{3}{2}\eta^2, \quad \alpha_2 = 2\eta, \quad \alpha_3 = 1, \quad \alpha_4 = 0,$$

obtained by putting $n = 0$, $m = \frac{1}{P_3}$ in (70). If we put $m = 0$, $n = \frac{1}{k}$, we find for C_β

$$\beta_1 = P_3 + \frac{3}{2}k\eta + \frac{3}{4}\eta^3, \quad \beta_2 = k + \frac{3}{2}\eta^2, \quad \beta_3 = \frac{3}{2}\eta, \quad \beta_4 = 1.$$

But (α_k, β_k) are simultaneous solutions of the equation

$$\frac{d\beta}{d\eta} = \frac{3}{2}\alpha,$$

which proves that *the ruled surface, which we are considering, is a developable, whose edge of regression is the twisted cubic C_β.*

The curve C_α is a conic, the intersection of the developable of the cubic with the osculating plane of C_y. Its equations are

$$x_4 = 0, \quad -3x_2^2 + 8x_1x_3 - 8kx_3^2 = 0.$$

We notice that *for $\Theta_4 = 0$, it coincides with the osculating conic*, a further interpretation of this condition. *In general, the two conics have a contact of the third order at P_y.*

If, in these equations, we change k into $-k$ we obtain the developable, cubic and conic associated with the second directrix of our congruence. A considerable number of other configurations are suggested by the combinations of these various curves and surfaces. I will refrain, however, from any further study in this direction.

The curve C_y is an asymptotic curve upon every derived ruled surface of the second kind. Moreover, the most general derived ruled surface of the second kind depends upon one arbitrary function, as does also the most general ruled surface containing C_y as an asymptotic curve. It is easy to see that the derived ruled surface of the second kind may be made to coincide with any ruled surface, upon which C_y is an asymptotic curve, if the independent variable be properly chosen.

Upon the derived ruled surface of the third kind, C_y can never be an asymptotic curve. It may, however, be one branch of the flecnode curve. In fact, if we form the quantities u_{ik} of the theory of ruled surfaces for system (27), we find

$$u_{12} = -\frac{P_2'}{P_2}.$$

But $u_{12} = 0$ is the condition that C_y may be a branch of the flecnode curve on the surface generated by P_yP_σ. Suppose that the variable has been so chosen as to make $P_2' = 0$. The most general transformation, which leaves this relation invariant, according to (11), satisfies the condition

$$-2\eta P_2 - \frac{5}{6}\mu' + \frac{5}{8}\eta\mu = 0$$

or

$$(71) \qquad -5\eta'' - 12\eta P_2 + 15\eta\eta' - 5\eta^3 = 0,$$

a differential equation of the second order for η. Moreover, two different solutions of this equation always give rise to two distinct ruled surfaces. For, let η_1 and η_2 be two such solutions, and let σ_1, σ_2 be the corresponding values of σ. Then, according to (9),

$$\sigma_\varkappa = \frac{1}{(\xi_\nu')^3}\left[\sigma + \frac{3}{2}\eta_\varkappa\varrho + \left(\mu_\varkappa + \frac{3}{2}\eta_\nu^2\right)z + \frac{1}{4}(\mu_\nu' + 4\eta_\varkappa\mu_\varkappa + 3\eta_\varkappa^3)y\right],$$
$$(\varkappa = 1, 2).$$

But, if the same ruled surface corresponds to η_1 and η_2, the three points y, σ_1 and σ_2 must be collinear. We must, therefore, be able to reduce

$$(\xi_1')^3\sigma_1 - (\xi_2')^3\sigma_2 = \frac{3}{2}(\eta_1 - \eta_2)\varrho + \cdots$$

to a multiple of y. But clearly, this is possible only if $\eta_2 = \eta_1$.

We see, therefore, that *there are ∞^2 derived ruled surfaces of the third kind upon which C_y is one branch of the flecnode curve.*

If P_2' is not zero, our problem leads to the differential equation

$$(72) \qquad \frac{d}{dx}\left(\eta' - \frac{1}{2}\eta^2\right) = \frac{6}{5}P_2' - \frac{12}{5}P_2\eta + 2\eta\left(\eta' - \frac{1}{2}\eta^2\right),$$

which is of the second order and of the third degree.

We have seen, in chapter XII, that the most general ruled surface, which has C_y as one branch of its flecnode curve, contains an arbitrary function in its general expression. We have also seen that, together with any such surface, its flecnode surface and each member of a single infinity of surfaces determined by these two, also contains C_y as one branch of its flecnode curve. One might imagine that there could be based upon these theorems a transformation theory of equation (72). This is not the case however. For, if one of the surfaces containing C_y as a branch of its flecnode curve is a derived ruled surface of the third kind, its flecnode surface is not, nor is any member of the family of ruled surfaces just mentioned.

Corresponding to the ∞^2 solutions of (72), or of

$$\mu' = \frac{6}{5}P_2' - \frac{12}{5}P_2\eta + 2\eta\mu,$$

we find ∞^2 positions for $P_{\bar\sigma}$, viz.:

$$4(\xi')^3\bar\sigma = \left(\frac{6}{5}P_2' - \frac{12}{5}P_2\eta + 6\eta\mu + 3\eta^3\right)y$$
$$+ (4\mu + 6\eta^2)z + 6\eta\varrho + 4\sigma.$$

The locus of these points is a cubic surface

$$(73)\quad 27P_2'x_4^3 - 36P_2x_3x_4^2 + 90x_2x_3x_4 - 90x_1x_4^2 - 40x_3^3 = 0,$$

which contains P_yP_z, the tangent of C_y, as a double line. It is, therefore, a ruled surface. It is a surface of the kind known as Cayley's cubic scroll.

If one derived surface of the third kind is known, upon which C_y is a branch of the flecnode curve, two others may be found by merely solving a quadratic equation.

In fact, suppose that a solution η of (72) be known. We may make a transformation of the independent variable, $\xi = \xi(x)$ such that

$$\frac{\xi''}{\xi'} = \eta.$$

In the resulting equation $\bar{P}_2' = 0$. If we again denote the independent variable by x, (72) becomes

$$\eta'' - \eta\eta' = -\frac{12}{5}P_2\eta + 2\eta\left(\eta' - \frac{1}{2}\eta^2\right),$$

where P_2 is a constant, since $P_2' = 0$. But we may satisfy this equation by putting $\eta = \text{const.}$, which gives the equation

$$\eta^3 + \frac{12}{5}P_2\eta = 0,$$

whence

$$\eta = 0, \quad \pm\sqrt{-\frac{12}{5}P_2}.$$

The root $\eta = 0$ gives the original solution. The other two are new.

§ 7. The principal tangent plane of two space curves. The covariants. Transition to Halphen's investigations.

Halphen has introduced a very important notion, which we shall now proceed to explain.

Let there be given two space curves having at a point P a contact of the n^{th} order. If these curves be projected from any center Q upon a plane, the projections will also have, in general, a contact of the n^{th} order at the point corresponding to P. *Halphen* shows that there exists a plane, passing through the common tangent of the two curves, such that if the center of projection be taken anywhere within it, the contact of the projections will be of order higher than n. This plane he calls the *principal tangent plane* of the two curves.[1])

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1) *Halphen*, Sur les invariants différentiels des courbes gauches. Journal de l'École Polytechnique, t. XXVIII (1880), p. 25.

We shall follow *Halphen* in determining the principal tangent plane, at P_y, of the curve C_y and its osculating cubic This will lead us to an especially simple form for the development of the equations of the curve, which is also due to *Halphen*, and on the basis of which he draws his further conclusions. It will also enable us to substitute for our system of covariants C_2, C_3, C_4 another system, whose geometrical significance will be apparent, and in terms of which C_2, C_3, C_4 may be expressed.

Assuming $P_2 = 0$, the equations of the osculating cubic, referred to the tetrahedron $P_y P_z P_\varrho P_\sigma$, are

$$3x_2x_4 - 2x_3^2 = 0, \quad 2x_3(5x_1 - 3P_3x_4) - 5x_2^2 = 0$$

Let us put

(74) $$\bar{x}_1 = x_2, \quad \bar{x}_2 = \frac{2}{5}x_3, \quad \bar{x}_3 = \frac{6}{25}x_4, \quad \bar{x}_4 = 5x_1 - 3P_3x_4,$$

and

(75) $$x = \frac{\bar{x}_1}{\bar{x}_4}, \quad y = \frac{\bar{x}_2}{\bar{x}_4}, \quad z = \frac{\bar{x}_3}{\bar{x}_4}.$$

Then the equations of the cubic reduce to

(76) $$y = x^2, \quad z = x^3.$$

The relation of the new tetrahedron of reference to the cubic is especially simple. The plane $\bar{x}_3 = 0$ is the osculating plane at P; $\bar{x}_2 = 0$ is some other plane through the tangent; this plane intersects the cubic in another point Q; the plane tangent to the cubic at Q and passing through P is $\bar{x}_1 = 0$; the osculating plane at Q is $\bar{x}_4 = 0$. Since the plane $\bar{x}_2 = 0$ may be chosen in an infinity of ways, in accordance with these conditions, we see that the reduction of the equations of the osculating cubic to the form (76) may be accomplished in an infinity of ways

For the curve C_y we have

(77)
$$\begin{aligned}
y_1 &= 1 - \frac{P_3}{3!}x^3 - \frac{P_4}{4!}x^4 - \frac{P_4'}{5!}x^5 - \frac{P_4'' - 4P_3^2}{6!}x^6 \\
&\quad - \frac{P_4^{(3)} - 5P_3P_4 - 12P_3P_3'}{7!}x^7 + \cdots, \\
y_2 &= x - \frac{P_3}{3!}x^4 - \frac{P_4 + 4P_3'}{5!}x^5 - \frac{2P_4' + 4P_3''}{6!}x^6 \\
&\quad + \frac{16P_3^2 - 3P_4'' - 4P_3^{(3)}}{7!}x^7 + \cdots, \\
y_3 &= \frac{1}{2}x^2 - \frac{4P_3}{5!}x^5 - \frac{1}{6!}(P_4 + 8P_3')x^6 \\
&\quad - \frac{1}{7!}(3P_4' + 12P_3'')x^7 + \cdots, \\
y_4 &= \frac{1}{6}x^3 - \frac{4}{6!}P_3x^6 - \frac{1}{7!}(P_4 + 12P_3')x^7 + \cdots.
\end{aligned}$$

If we refer it to the same tetrahedron, to which we have just referred the osculating cubic, and if we put

$$\xi = \frac{\bar{y}_1}{\bar{y}_4}, \quad \eta = \frac{\bar{y}_2}{\bar{y}_4}, \quad \zeta = \frac{\bar{y}_3}{\bar{y}_4},$$

we shall find

$$(78) \qquad \begin{aligned} \eta &= \xi^2 + \frac{125}{36}(8P_3' - 5P_4)\xi^6 + \frac{625}{7\cdot 36}(8P_3'' - 5P_4')\xi^7 + \cdots, \\ \zeta &= \xi^3 - \frac{25}{3}P_3\xi^6 + \frac{125}{7\ 12}(36P_3' - 25P_4)\xi^7 + \cdots. \end{aligned}$$

If we put

$$(79) \qquad \xi = \varepsilon X, \quad \eta = \varepsilon^2 Y, \quad \zeta = \varepsilon^2 Z, \quad \varepsilon^3 = -\frac{3}{25P_3},$$

assuming, therefore, that P_3 is not zero, this becomes

$$(80) \qquad \begin{aligned} Y &= X^2 + \lambda_6 X^6 + \lambda_7 X^7 + \cdots, \\ Z &= X^3 + \mu_6 X^6 + \mu_7 X^7 + \cdots, \end{aligned}$$

where

$$(81) \qquad \begin{aligned} \lambda_6 &= \frac{125}{36}(8P_3' - 5P_4)\varepsilon^4, & \lambda_7 &= \frac{625}{7\ 36}(8P_3'' - 5P_4')\varepsilon^5, \\ \mu_6 &= 1, & \mu_7 &= \frac{125}{7\ 12}(36P_3' - 25P_4)\varepsilon^4. \end{aligned}$$

But we can make a further change of coordinates without disturbing the form (80) of the development. For, as we remarked above, the plane $\bar{x}_2 = 0$ may be any plane through the tangent. It may be taken in such a way as to make λ_6 vanish, as we shall presently see. *The plane thus obtained will obviously be the principal tangent plane of the curve C_y and its osculating cubic.*

Instead of working out the transformation geometrically, we shall put with Halphen

$$(82a) \qquad \bar{X} = \frac{\xi_1}{\omega_1}, \quad \bar{Y} = \frac{\eta_1}{\omega_1}, \quad \bar{Z} = \frac{Z}{\omega_1},$$

where

$$(82b) \qquad \begin{aligned} \omega_1 &= 1 + 3pX + 3p^2Y + p^3Z, \quad \eta_1 = Y + pZ, \\ \xi_1 &= X + 2pY + p^2Z, \end{aligned}$$

the quantity p being arbitrary.

We shall then have

$$\begin{aligned} \omega_1 &= (1 + pX)^3 + p^2(3\lambda_6 + p\mu_6)X^6 + \cdots, \\ \eta_1 &= X^2(1 + pX) + (\lambda_6 + p\mu_6)X^6 + (\lambda_7 + p\mu_7)X^7 + \cdots, \\ \xi_1 &= X(1 + pX)^2 + p(2\lambda_6 + p\mu_6)X^6 + p(2\lambda_7 + p\mu_7)X^7 + \cdots, \\ \omega_1\eta_1 - \xi_1^2 &= (\lambda_6 + p\mu_6)X^6 + [\lambda_7 + p\mu_7 + p(p\mu_6 - \lambda_6)]X^7 + \cdots, \\ \omega_1^2 Z - \xi_1^3 &= \mu_6 X^6 + (6p\mu_6 + \mu_7)X^7 + \cdots. \end{aligned}$$

On the other hand, $\frac{X^6}{\omega_1^2}$ and $\left(\frac{\xi_1}{\omega_1}\right)^6$ coincide up to terms of the eleventh order, while

$$\frac{X^6}{\omega_1^8} = \left(\frac{\xi_1}{\omega_1}\right)^3 - 3p\left(\frac{\xi_1}{\omega_1}\right)^7 + \cdots.$$

We shall find, therefore,

$$\begin{aligned} Y &= \overline{X}^2 + \Lambda_6 \overline{X}^6 + \Lambda_7 \overline{X}^7 + \cdots, \\ \overline{Z} &= \overline{X}^3 + M_6 \overline{X}^6 + M_7 \overline{X}^7 + \cdots, \end{aligned} \tag{83}$$

where

$$\begin{aligned} M_6 &= \mu_6, & M_7 &= \mu_7 + 3p\mu_6, \\ \Lambda_6 &= \lambda_6 + p\mu_6, & \Lambda_7 &= \lambda_7 + p\mu_7 + p'(p\mu_7 - \lambda_6). \end{aligned} \tag{84}$$

For

$$p = -\frac{\lambda_6}{\mu_6},$$

this reduces to the canonical form required, in which we may also assume $\mu_6 = 1$, if the curve does not belong to a linear complex.

If we combine the various transformations, which we have made successively, we get the following result *If we introduce non-homogeneous coordinates by putting*

$$X = \frac{\bar{y}_1}{\bar{y}_4}, \quad Y = \frac{\bar{y}_2}{\bar{y}_4}, \quad Z = \frac{\bar{y}_3}{\bar{y}_4}, \tag{85a}$$

where

$$\begin{aligned} y_1 &= 25\varepsilon^2 y_2 + 20p\varepsilon y_3 + 6p^2 y_4, \\ y_2 &= \phantom{25\varepsilon^2 y_2 + {}} 10\varepsilon y_3 + 6p y_4, \\ y_3 &= \phantom{25\varepsilon^2 y_2 + 20p\varepsilon y_3 + {}} 6y_4, \\ y_4 &= 125\varepsilon^3 y_1 + 75p\varepsilon^2 y_2 + 30p^2\varepsilon y_3 + (6p^3 - 75P_3\varepsilon^3)y_4, \end{aligned} \tag{85b}$$

and where

$$\varepsilon^3 = -\frac{3}{25P_3}, \quad p = -\frac{125}{36}(8P_3' - 5P_4)\varepsilon^4, \tag{85c}$$

the development of the equations of the curve C_y may be written in the canonical form

$$\begin{aligned} Y &= X^2 + \Lambda_7 X^7 + \cdots, \\ Z &= X^3 + X^6 + M_7 X^7 + \cdots, \end{aligned} \tag{86}$$

where Λ_7, $M_7, \ldots$ are given by (84) together with (81). This transformation is valid if C_y does not belong to a linear complex, and can obviously be made in three different ways

The coefficients of (86) are absolute invariants. We find, in fact,

$$\begin{aligned} \Lambda_7 &= \frac{5^4\varepsilon^6}{2^4\ 3^3\ 7\,\Theta_3}\left(25\,\Theta_4^2 + 20\,\Theta_8 - 4\,\Theta_{3\ 1}\right), \\ M_7 &= \frac{5^4\varepsilon^4}{2\ 3\ 7}\Theta_4, \end{aligned} \tag{87}$$

where

$$\Theta_8 = 4\Theta_4\Theta_3' - 3\Theta_3\Theta_4'. \tag{88}$$

From (86) and (76) it is clear that the plane $Y = 0$ or $y_2 = 0$ is the principal plane of the curve C_y and its osculating cubic. In the original system of coordinates its equation will, therefore, be

$$(89) \qquad 4P_3 x_3 + (8P_3' - 5P_4)x_4 = 0.$$

If C_y belongs to a linear complex, it coincides with the osculating plane.

The point, which corresponds to (89) in the osculating linear complex, is

$$(90) \qquad (5P_4 - 8P_3')y + 4P_3 z,$$

or, in invariant form,

$$(91) \qquad \left(\Theta_3' + \frac{5}{2}\Theta_4\right)y + 2\Theta_3 z,$$

an expression whose covariance may be verified directly. We shall call this point the *principal point of the tangent*. We thus obtain a curve on the developable of C_y, which may be called *its principal curve.*

If we make a transformation such that

$$(92) \qquad 3\Theta_3 \eta = \Theta_3' + \frac{5}{2}\Theta_4,$$

the point $P_{\bar{z}}$ describes the principal curve on the developable of C_y. The points $P_{\bar{\varrho}}$ and $P_{\bar{\sigma}}$ also describe perfectly definite curves, whose expressions may be obtained from (9) by substituting for η the expression (92). *Any covariant may be expressed in terms of y and the three which we have just determined, which may therefore serve to replace the covariants C_2, C_3 and C_4.*

If C_y belongs to a linear complex, these four curves all coincide, so that a different set of fundamental covariants must then be selected.

In this exceptional case our fundamental tetrahedron $P_y P_z P_\varrho P_\sigma$ gives rise to a most remarkable configuration. If we put again $P_2 = 0$, we have in this case also $P_3 = 0$. $P_y P_\sigma$ generates one of the developables of the third kind of which C_σ is the cuspidal edge, while $P_y P_z$ of course generates the developable of which C_y is the cuspidal edge. The surface generated by $P_y P_\varrho$ is not developable. Its equations become

$$\varrho'' + P_4 y = 0, \quad y'' - \varrho = 0.$$

It belongs to the same linear complex as C_y, and C_y and C_ϱ are the two branches of its complex curve, which is at the same time an asymptotic curve. $P_z P_\varrho$ generates a developable, since

$$z' = \varrho,$$

of which C_z is the cuspidal edge. $P_\varrho P_\sigma$ generates a developable, since

$$\varrho' = \sigma,$$

of which C_ϱ is the cuspidal edge. Finally $P_z P_\sigma$ generates a ruled surface whose equations are

$$\sigma'' - \frac{P_4'}{P_4}\sigma' + P_4 z = 0, \quad z'' - \sigma = 0,$$

upon which C_z and C_σ are asymptotic lines.

We find, therefore, the following theorem.

If the curve belongs to a linear complex we may, in an infinity of ways, choose the fundamental tetrahedron so that four of its edges give rise to developables, whose cuspidal edges are described by the four vertices. The other two edges of the tetrahedron will then give rise to ruled surfaces, upon each of which the vertices of the tetrahedron trace a pair of asymptotic curves. The latter coincide with the two branches of the complex curve for the derived surface of the second kind.

As in the case of differential equations of the third order, we conclude from the canonical development obtained in this chapter: *the necessary and sufficient conditions for the equivalence of two linear differential equations of the fourth order are the equality of the corresponding absolute invariants.*

Examples.

Ex. 1. Examine the conditions for the existence of conics or space cubics upon the developable whose cuspidal edge is C_y

Ex. 2. Discuss the cases in which the derivative ruled surface of the second kind belongs to a linear complex or congruence.

Ex. 3.* The osculating cubics of C_y form a surface. Do there exist such curves C_y whose osculating cubics are asymptotic curves upon this surface? If there are such, determine them and find the second family of asymptotic lines.

Ex. 4.* Consider the same problem for the surface of torsal cubics.

Ex. 5. Assuming $p_1 = P_2 = 0$, deduce the differential equation of the sixth order for the line-coordinates of the tangents of C_y. (Halphen.)

Ex. 6. Find the conditions that the principal curve of the developable of C_y may be a conic, a space cubic, or a curve belonging to a linear complex.

CHAPTER XIV.

PROJECTIVE DIFFERENTIAL GEOMETRY OF SPACE CURVES (CONTINUED).

§ 1. Introduction of Halphen's differential invariants, and identification with the invariants of the preceding chapter.

Let x, y, z be the (non-homogeneous) coordinates of a point m, and let y and z be given as functions of x, so that the equations

$$y = \varphi(x), \quad z = \psi(x)$$

represent a curve (m). All differentiations are to be taken with respect to x, if no other independent variable is specified.

Put

$$u = \frac{1}{12}\left(y'' z^{(3)} - z'' y^{(3)}\right) \tag{1}$$

Then, clearly, the equation $u = 0$ is characteristic of a plane curve, if satisfied identically In general the values of x, for which u vanishes, give the points whose osculating plane hyperosculates the curve. We write with *Halphen*,

$$\begin{aligned} a_n &= \frac{1}{4 \cdot 5 \cdots n} \frac{y'' z^{(n)} - y^{(n)} z''}{y'' z^{(3)} - y^{(3)} z''}, \\ b_n &= \frac{1}{3 \cdot 4 \cdots n} \frac{y^{(n)} z^{(3)} - y^{(3)} z^{(n)}}{y'' z^{(3)} - y^{(3)} z''}, \end{aligned} \qquad (n = 4, 5, 6, \ldots). \tag{2}$$

Suppose that the point m, or (x, y, z), is an ordinary point of the curve (m); let X, Y, Z be the coordinates of a point of the curve, in the vicinity of m. Then

$$Y = y + y'(X - x) + \frac{1}{2} y''(X - x)^2 + \frac{1}{3!} y^{(3)}(X - x)^3 + \cdots,$$

$$Z = z + z'(X - x) + \frac{1}{2} z''(X - x)^2 + \frac{1}{3!} z^{(3)}(X - x)^3 + \cdots.$$

Make the following transformation of coordinates:

$$\begin{aligned} X_1 &= X - x, \\ Y_1 &= 2\,\frac{z^{(3)}[Y - y - y'(X - x)] - y^{(3)}[Z - z - z'(X - x)]}{y'' z^{(3)} - y^{(3)} z''}, \\ Z_1 &= 6\,\frac{y''[Z - z - z'(X - x)] - z''[Y - y - y'(X - x)]}{y'' z^{(3)} - y^{(3)} z''}. \end{aligned} \tag{3}$$

This is clearly a projective transformation. The developments of the equations of the curve assume the simpler form

$$Z_1 = X_1^3 + a_4 X_1^4 + a_5 X_1^5 + \cdots,$$
$$Y_1 = X_1^2 + b_4 X_1^4 + b_5 X_1^5 + \cdots,$$

where a_n, b_n are precisely the quantities defined by (2). We may always make the projective transformation (3), if the point m is not one whose osculating plane hyperosculates the curve, which case we shall suppose excluded. We may, therefore, assume that the equations have been written, in the first place, in the form

$$(4) \qquad \begin{aligned} z &= x^3 + a_4 x^4 + a_5 x^5 + \cdots, \\ y &= x^2 + b_4 x^4 + b_5 x^5 + \cdots. \end{aligned}$$

An equation, which expresses a projective property of the curve, must, therefore, be an equation between the quantities a_n, b_n. It is in terms of them, that *Halphen* has expressed his differential invariants.

If the curve belongs to a linear complex, there must be verified an equation of the form

$$A + By' + Cz' + D(xy' - y) + E(xz' - z) + F(yz' - y'z) = 0,$$

where $A, \ldots F$ are constants.[1]) If we substitute (4) for y and z, and equate to zero the coefficients of $x^0, x^1, x^2, \ldots x^5$, we find

$$A = 0, \quad B = 0, \quad 3C + D = 0, \quad 4a_4 C + 2E = 0,$$
$$5a_5 C + 3b_4 D + 3a_4 E + F = 0,$$
$$6a_6 C + 4b_5 D + 4a_5 E + 2a_4 F = 0.$$

Eliminating $A, \ldots F$ gives $r = 0$, where

$$(5) \qquad r = a_6 - 2b_5 - 3a_4 a_5 + 3a_4 b_4 + 2a_4^3.$$

The equation $r = 0$ is the condition that the curve (m) may belong to a linear complex.

We proceed to reduce (4) to its canonical form by Halphen's method. The geometry of this reduction has already been explained, so that it will suffice to give the transformation in its analytical form. Put

$$(6) \qquad \xi = x + my + nz, \quad \omega = 1 + Ax + By + Cz$$

and develop the quantities $\omega y, \omega^2 z, \xi^2, \xi^3$ up to terms of the seventh order inclusively. Form the differences

$$\omega y - \xi^2 \quad \text{and} \quad \omega^2 z - \xi^3,$$

and dispose of the five unknown quantities m, n, A, B, C in such a way that the terms up to and including the fifth order shall vanish. Then we shall have

1) Cf. chapter VII.

$$\omega y - \xi^2 = \lambda_6 x^6 + \lambda_7 x^7 + \cdot \quad ,$$
$$\omega^2 z - \xi^3 = \mu_6 x^6 + \mu_7 x^7 +$$

On the other hand

$$\frac{x^6}{\omega^2} = \left(\frac{\xi}{\omega}\right)^6 + (4A - 6m)\left(\frac{\xi}{\omega}\right)^7 + \quad ,$$
$$\frac{x^6}{\omega^3} = \left(\frac{\xi}{\omega}\right)^6 + (3A - 6m)\left(\frac{\xi}{\omega}\right)^7 + \quad .$$

If, therefore, we put

$$X = \frac{\xi}{\omega}, \quad Y = \frac{y}{\omega}, \quad Z = \frac{z}{\omega},$$

we shall find

$$(7) \qquad \begin{aligned} Y &= X^2 + \lambda_6 X^6 + \lambda_7 X^7 + \quad , \\ Z &= X^3 + \mu_6 X^6 + \mu_7 X^7 + \quad . \end{aligned}$$

The calculations are clearly indicated. For the details, the reader may consult Halphen's memoir.[1]) We find, in this way, the values

$$(8) \qquad \begin{aligned} &A = -2a_4, \quad B = 3a_4^2 + 3b_4 - 2a_5, \quad C = -b_5, \\ &m = -a_4, \quad\quad n = a_4^2 + 2b_4 - a_5 \end{aligned}$$

for the coefficients of the transformation (6). As a consequence

$$(9) \qquad \begin{aligned} &\mu_6 = a_6 - 2b_5 - 3a_4a_5 + 3a_4b_4 + 2a_4^3 = v, \\ &\lambda_6 = b_6 - a_4b_5 - 4a_5b_4 + 4a_4^2b_4 - 2a_4^2a_5 + 2b_4^2 + a_5^2 + a_4^4 = w \end{aligned}$$

and

$$(10) \qquad \begin{aligned} &\mu_7 = a_7 + 3b_4^2 - 4b_4a_5 + 3b_5a_4 - 4a_6a_4 + 6a_4^2a_5 - 3a_4^4, \\ &\lambda_7 = l - 2a_4\lambda_6, \quad \text{where} \\ &l = b_7 + 5b_4b_5 - 5a_5b_5 + 4a_4^2b_5 \\ &\quad - 2(a_4^2 + 2b_4 - a_5)(a_6 + a_4^3 + a_4b_4 - 2a_4a_5). \end{aligned}$$

We have reduced the development of the equations of the curve to the form (7). In § 7 of chapter XIII, we have shown how to set up a transformation, which involves an arbitrary constant p, and does not disturb this form of the development We put

$$(11) \quad \omega_1 = 1 + 3pX + 3p^2Y + p^3Z, \quad \eta_1 = Y + pZ, \quad \xi_1 = X + 2pY + p^2Z,$$

and

$$(11\text{a}) \qquad X = \frac{\xi_1}{\omega_1}, \quad Y = \frac{\eta_1}{\omega_1}, \quad Z = \frac{Z}{\omega_1},$$

whence

$$(12) \qquad \begin{aligned} \overline{Y} &= \overline{X}^2 + A_6\overline{X}^6 + A_7\overline{X}^7 + \quad , \\ \overline{Z} &= \overline{X}^3 + M_6\overline{X}^6 + M_7\overline{X}^7 + \cdot \quad , \end{aligned}$$

where

1) *Halphen*, Journal de l'École Polytechnique, t XXVIII (1880), pp. 30 et sequ.

$$(13)\qquad \begin{aligned} M_6 &= \mu_6, & M_7 &= \mu_7 + 3p\mu_6, \\ \Lambda_6 &= \lambda_6 + p\mu_6, & \Lambda_7 &= \lambda_7 + p\mu_7 + p(p\mu_6 - \lambda_6). \end{aligned}$$

If we put

$$p = -\frac{\lambda_6}{\mu_6} = -\frac{w}{v},$$

we find the form

$$(14)\qquad \begin{aligned} Y &= X^2 + \beta_7 X^7 + \beta_8 X^8 + \cdots, \\ Z &= X^3 + \alpha_6 X^6 + \alpha_7 X^7 + \alpha_8 X^8 + \cdots, \end{aligned}$$

where

$$\alpha_6 = \mu_6 = v, \quad \alpha_7 = \mu_7 - 3\lambda_6, \quad \beta_7 = \frac{\mu_6\lambda_7 - \lambda_6\mu_7 + 2\lambda_6^2}{\mu_6},$$

from which the canonical form may be derived at once.

It is obvious that the coefficients α_k and β_k are relative invariants. We thus find *Halphen's fundamental invariants of the seventh order*

$$(15)\qquad s_7 = \alpha_7 = \mu_7 - 3\lambda_6, \quad t_7 = \mu_6\beta_7 = \mu_6\lambda_7 - \lambda_6\mu_7 + 2\lambda_6^2.$$

From s_7, t_7 and v we form other invariants by the Jacobian process:

$$(16)\qquad \begin{aligned} s_8 &= \frac{1}{8}\left(v s_7' - \frac{4}{3} v' s_7\right), \quad t_8 = \frac{1}{8}\left(v t_7' - \frac{8}{3} v' t_7\right), \\ s_9 &= \frac{1}{9}\left(v s_8' - \frac{8}{3} v' s_8\right), \quad \text{etc.} \end{aligned}$$

Halphen speaks of these as the *fundamental invariants.* But the coefficients α_k, β_k, of (14), form a second series of invariants which Halphen speaks of as *canonical invariants*

We shall need the equations between s_8, t_8, α_8 and β_8 In order to find them, we recur to the definitions of the quantities a_n and b_n. We find directly

$$(17)\qquad \begin{aligned} a_n' &= (n+1)a_{n+1} - 3b_n - 4a_4a_n, \\ b_n' &= (n+1)b_{n+1} - 4b_4a_n. \end{aligned}$$

We shall make use of these equations to express s_8 and t_8 in terms of the canonical invariants, putting

$$\begin{aligned} a_4 &= a_5 = 0, & a_6 &= \alpha_6, \ldots a_n = \alpha_n, \\ b_4 &= b_5 = b_6 = 0, & b_7 &= \beta_7, \ldots b_n = \beta_n \end{aligned}$$

after the differentiations. Thus we find

$$\begin{aligned} v &= \alpha_6, & s_7 &= \alpha_7, & t_7 &= \beta_7\alpha_6, \\ v' &= 7\alpha_7, & s_7' &= 8(\alpha_8 - 3\beta_7), & t_7' &= 4(2\alpha_6\beta_8 + 3\alpha_6^3). \end{aligned}$$

Therefore, according to (16),

$$(18)\qquad \begin{aligned} s_8 &= \alpha_6(\alpha_8 - 3\beta_7) - \frac{7}{6}\alpha_7^2, \\ t_8 &= \alpha_6^2\beta_8 - \frac{7}{3}\alpha_6\alpha_7\beta_7 + \frac{3}{2}\alpha_6^4, \end{aligned}$$

whence

(19)
$$v\alpha_8 = s_8 + 3t_7 + \frac{7}{6}s_7^2,$$
$$v^2\beta_8 = t_8 + \frac{7}{3}s_7t_7 - \frac{3}{2}v^4,$$

the required equations.

It remains to find the relations between these *differential invariants of Halphen*, and the invariants of chapter XIII.

Since $\Theta_3 = 0$, as well as $v = 0$ express the condition for a curve belonging to a linear complex, Θ_3 and v can differ only by a constant factor. We may determine this factor by means of any special curve for which v does not vanish. Consider the equation

$$y^{(4)} - \frac{1}{x}y^{(3)} = 0,$$

for which

$$y_1 = 1, \quad y_2 = x, \quad y_3 = x^2, \quad y_4 = x^4$$

form a fundamental system. We find

$$\Theta_3 = -\frac{15}{32}x^{-3}.$$

On the other hand, introduce non-homogeneous coordinates, by putting

$$\xi = \frac{y_2}{y_1}, \quad \eta = \frac{y_3}{y_1}, \quad \zeta = \frac{y_4}{y_1}.$$

Then

$$\xi = x, \quad \eta = \xi^2, \quad \zeta = \xi^4,$$

whence

$$a_4 = \frac{1}{4x}, \quad a_5 = a_6 = b_4 = b_5 = 0,$$

so that

$$v = \frac{1}{32}x^{-3}.$$

We have, therefore,

(20)
$$\Theta_3 = -15v.$$

In order to derive the canonical form from (14), we put

$$X = a\bar{X}, \quad Y = a^2\bar{Y}, \quad Z = a^3\bar{Z}, \quad a^3 = v.$$

In fact, (14) will then become identical with (86) of chapter XIII. Since we have there

$$\varepsilon^3 = -\frac{3}{25\Theta_3},$$

we may put

$$a\varepsilon = \frac{1}{5}.$$

Identifying the two developments, we find

(21)
$$s_7 = \frac{1}{42}\Theta_4, \quad t_7 = -\frac{25\Theta_4^2 + 20\Theta_8 - 4\Theta_3 \cdot 1}{2^4\cdot 3^4\cdot 5^2 \cdot 7},$$

whence

(21a)
$$s_8 = \frac{\Theta_8}{2^4\cdot 3^3\cdot 5\cdot 7}.$$

§ 2. The osculating quadric surface.

A quadric surface is determined by nine points. If we pass a quadric through nine points of the curve, and allow them to approach coincidence, we obtain, as a limit, the osculating quadric.

Let the equations of the curve be written in the form

$$(22)\qquad \begin{aligned} y &= x^2 + \beta_7 x^7 + \beta_8 x^8 + \cdots, \\ z &= x^3 + \alpha_6 x^6 + \alpha_7 x^7 + \alpha_8 x^8 + \cdots. \end{aligned}$$

Then we shall have

$$(23)\qquad \begin{aligned} y - x^2 &= \beta_7 x^7 + \beta_8 x^8 + \cdots, \\ z - xy &= \alpha_6 x^6 + \alpha_7 x^7 + (\alpha_8 - \beta_7) x^8 + \cdots, \\ xz - y^2 &= \alpha_6 x^7 + \alpha_7 x^8 + \cdots. \end{aligned}$$

Consider, now, the quadric surface, whose equation is

$$(24)\qquad \begin{vmatrix} y - x^2, & z - xy, & xz - y^2, & z^2 \\ 0, & \alpha_6, & 0, & 1 \\ \beta_7, & \alpha_7, & \alpha_6, & 0 \\ \beta_8, & \alpha_8 - \beta_7, & \alpha_7, & 0 \end{vmatrix} = 0.$$

If we wish to find its intersection with the curve, we substitute the values (22) for y and z. Clearly, the development of the determinant will begin with a term of the 9^{th} order in x. The quadric, therefore, intersects the curve for $x = 0$ in nine coincident points; i. e. (24) is the equation of the osculating quadric.

Expanding this determinant, we obtain

$$(25)\qquad \begin{aligned} &(\alpha_6 \beta_7 - \alpha_6 \alpha_8 + \alpha_7^2)[\alpha_6 (y - x^2) + \beta_7 (y^2 - xz)] \\ &\quad + (\alpha_6 \beta_8 - \alpha_7 \beta_7)[\beta_7 (y^2 - xz) + \alpha_6 (z - xy - \alpha_6 z^2)] = 0, \end{aligned}$$

or, upon introducing the fundamental invariants,

$$(26)\qquad \begin{aligned} &\left(s_8 + 2t_7 + \frac{1}{6} s_7^2\right)\left[v^2 (x^2 - y) + t_7 (xz - y^2)\right] \\ &\quad + \left(t_8 + \frac{4}{3} s_7 t_7 - \frac{3}{2} v^4\right)\left[s_7 (y^2 - xz) + v (z - xy - v z^2)\right] = 0. \end{aligned}$$

This equation ceases to be valid if $v = 0$, i. e. if the curve belongs to a linear complex. In this case we adopt a different canonical form. The intermediate form (7) becomes

$$\begin{aligned} Y &= X^2 + \lambda_6 X^6 + \lambda_7 X^7 + \lambda_8 X^8 + \cdots, \\ Z &= X^3 + \mu_7 X^7 + \mu_8 X^8 + \cdots, \end{aligned}$$

since $\mu_6 = 0$ in this case. By means of transformation (11), this assumes the form (12). We may make $A_7 = 0$, by choosing p subject to the condition

$$\lambda_7 + p(\mu_7 - \lambda_6) = 0.$$

The canonical form, thus obtained, is

$$(27)\qquad \begin{aligned} y &= x^2 + A_6 x^6 + A_8 x^8 + \quad, \\ z &= x^3 + M_7 x^7 + M_8 x^8 + \quad, \end{aligned}$$

so that

$$(28)\qquad (2A_6 - M_7)(y - x^2 - A_6 z^2) - A_8 (y^2 - xz) = 0$$

is the osculating quadric. We notice that the plane $y = 0$ is tangent to the quadric at the origin. The tetrahedron of reference, for which the development assumes the canonical form, is therefore determined by the osculating plane, osculating cubic curve, and osculating quadric surface. *The vertices of this tetrahedron furnish a complete system of covariants, for the case of a curve belonging to a linear complex,* if the curve is not a space cubic. For, this tetrahedron, can degenerate only if the osculating plane and the plane tangent to the osculating quadric coincide. This will be the case whenever

$$(29)\qquad s_8 + 2t_7 + \frac{1}{6} s_7^2 = 0$$

But if $\Theta_3 = 0$ at the same time, this gives also $\Theta_4 = 0$, i.e the curve can only be a space cubic.

If

$$(30)\qquad t_8 + \frac{4}{3} s_7 t_7 - \frac{3}{2} v^4 = 0,$$

the plane tangent to the osculating quadric, coincides with the principal plane of the curve and its osculating cubic.

If the two equations (29) and (30) are satisfied simultaneously, the osculating quadric (26) becomes indeterminate. Therefore, *the simultaneous equations* (29) *and* (30) *are characteristic of the biquadratic curves*

The biquadratic curves may be studied by means of elliptic functions, as were the plane cubics. The reader will find such a treatment in *Halphen*'s memoire, Sur les invariants différentiels des courbes gauches, p. 96 et sequ.

§ 3. Anharmonic curves.

If the absolute invariants of the differential equation of a space curve are constant, the curve is said to be *anharmonic.* It is easy to see, that the differential equation may then be transformed into one with constant coefficients. Moreover, the reduction of such an equation to its semi-canonical form leaves the coefficients constants. We may, therefore, assume that the differential equation has the form

(31) $$y^{(4)} + 6P_2 y'' + 4P_3 y' + P_4 y = 0,$$

where P_2, P_3, P_4 are constants. Let $r_1, \ldots r_4$ be the four roots, supposed distinct, of the equation

(32) $$r^4 + 6P_2 r^2 + 4P_3 r + P_4 = 0.$$

Then the functions

(33) $$y_k = e^{r_k x} \quad (k = 1, 2, 3, 4)$$

will form a fundamental system of (31), so that

$$y_2^{r_3-r_1} y_1^{r_2-r_3} = y_3^{r_2-r_1}, \quad y_2^{r_4-r_1} y_1^{r_2-r_4} = y_4^{r_2-r_1}.$$

If, therefore, we introduce non-homogeneous coordinates, by putting

$$X = \frac{y_2}{y_1}, \quad Y = \frac{y_3}{y_1}, \quad Z = \frac{y_4}{y_1},$$

we shall find, as the equations of the anharmonic curve,

(34) $$Y = X^\lambda, \quad Z = X^\mu,$$

where

(35) $$\lambda = \frac{r_3 - r_1}{r_2 - r_1}, \quad \mu = \frac{r_4 - r_1}{r_2 - r_1}.$$

The curve admits a one-parameter group of projective transformations into itself, viz:

(36) $$X = aX, \quad Y = a^\lambda Y, \quad Z = a^\mu Z,$$

where a is an arbitrary constant. By means of this transformation any point of the curve, which is not a vertex of the triangle of reference, may be converted into any other

From this theorem we conclude, as in the case of plane anharmonic curves: *the four points in which any tangent of the curve intersects the faces of the fundamental tetrahedron, and the point of contact, form a group of five points upon the tangent, which remains projective to itself as the point of contact moves along the curve.*

This theorem may also be verified by computing two of the double ratios of this group of five points. They will be found to be λ and μ respectively

The curve belongs to a so-called tetrahedral complex. In fact, its tangents intersect the tetrahedron of reference in four points whose cross-ratio is constant, and the totality of lines which are so related to a fixed tetrahedron constitute a tetrahedral complex.[1])

1) For curves and surfaces belonging to a tetrahedral complex, cf *Lie-Scheffers,* Geometrie der Berührungstransformationen, pp. 311—398.

From (32) we have

$$r_1 + r_2 + r_3 + r_4 = 0,$$
$$r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4 = 6P_2,$$
$$r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2 = -4P_3,$$
$$r_1 r_2 r_3 r_4 = P_4,$$

whence, together with (35), we find

$$P_2 = \frac{-r_1^2}{6(\lambda+\mu+1)^2}\left[6(\lambda^2+\mu^2+1) - 4(\lambda\mu+\lambda+\mu)\right],$$

$$(37) \qquad P_3 = \frac{2r_1^3}{(\lambda+\mu+1)^3}\left[\lambda^3+\mu^3-\lambda\mu^2-\lambda^2\mu-\lambda^2-\mu^2+2\lambda\mu-\lambda-\mu+1\right],$$

$$P_4 = \frac{r_1^4}{(\lambda+\mu+1)^3}(\lambda+\mu-3)(-3\lambda+\mu+1)(\lambda-3\mu+1).$$

Since the coefficients are constants, we have

$$\Theta_3 = P_3, \quad \Theta_4 = P_4, \quad \Theta_{3\cdot1} = -\frac{108}{5}P_2 P_3^2.$$

If, therefore, the absolute invariants of an anharmonic curve

$$(38) \qquad \frac{\Theta_4^3}{\Theta_3^4} = \alpha, \quad \frac{\Theta_{3\cdot1}^3}{\Theta_3^8} = \beta$$

are given, these equations will serve to determine the exponents λ and μ. The different values of λ and μ, which correspond to given values of the invariants α, β, correspond to the twenty-four permutations of the faces of the tetrahedron of reference (cf. the corresponding remark for plane anharmonics).

The cases, when two or more of the exponents r_k coincide, may be obtained from the general case as in the theory of plane anharmonics.

Equations (38) show that, to any anharmonic curve corresponds dualistically another with the same absolute invariants, since α and β contain only even powers of Θ_3.

Let $\Theta_3 = 0$, and let the absolute invariant $\frac{\Theta_{4\cdot1}^2}{\Theta_4^5}$ be a constant. Assuming $\Theta_4 \neq 0$, so that the curve is not a space cubic, we may choose the variables so as to have

$$p_1 = 0, \quad \Theta_4 = 1, \quad \text{whence } \Theta_{4\cdot1} = \text{const.}$$

We shall then have

$$y^{(4)} + 6P_2 y'' + P_4 y = 0,$$

where P_2 and P_4 are constants. Let

$$r_1, \; -r_1, \; r_2, \; -r_2$$

be the roots of

$$r^4 + 6P_2 r^2 + P_4 = 0,$$

and let

$$\eta_1 = e^{r_1}, \quad \eta_2 = e^{r_2}, \quad \eta_3 = e^{-r_1}, \quad \eta_4 = e^{-r_2}.$$

Then

$$\eta_1\eta_3 - \eta_2\eta_4 = 0,$$

i. e. the curve lies on a quadric surface. In other words: *any anharmonic curve, whose tangents belong to a linear complex, is on a quadric, and conversely.*

But the same reasoning applied to the adjoint equation shows further: *any anharmonic curve, whose tangents belong to a linear complex, is the cuspidal edge of a developable which is circumscribed about a quadric surface.*[1])

We shall leave it to the reader to prove; that these two quadrics upon one of which lies the curve, while its developable is circumscribed about the other, have a skew quadrilateral in common.

As in the case of plane curves, we may determine at any point of a given space curve, an anharmonic having with the given curve a contact of the seventh order Let v, s_7, t_7, etc. be the invariants of the given curve, $\bar v$, $\bar s_7$, $\bar t_7$ the corresponding invariants of the osculating anharmonic. $\bar s_8$, $\bar t_8$, etc. will be zero, while

$$(39)\qquad \frac{\bar s_7}{\bar v^{\frac{4}{3}}} = \frac{s_7^{(0)}}{v^{(0)\frac{4}{3}}}, \qquad \frac{\bar t_7}{\bar v^{\frac{8}{3}}} = \frac{t_7^{(0)}}{v^{(0)\frac{8}{3}}},$$

where $v^{(0)}$, $s_7^{(0)}$ and $t_7^{(0)}$ are the numerical values of v, s_7 and t_7 at the given point.

The absolute invariants of the osculating anharmonic being known, the problem arises: to determine its principal tetrahedron

In order to solve this problem we prove first, the following theorem due to *Fouret.*[2])

Let X, Y, Z, T be four linear functions of x, y, z, such as

$$(40)\qquad X = ax + a'y + a''z + a''', \text{ etc.,}$$

and consider the differential equations

$$(41)\qquad \frac{dx}{X - Tx} = \frac{dy}{Y - Ty} = \frac{dz}{Z - Tz}.$$

The general integral of these equations will be

$$(42)\qquad T_1^{r-1}Y_1 = cX_1^r, \quad T_1^{s-1}Z_1 = c'X_1^s,$$

where c and c' are arbitrary constants, and where X_1, Y_1, Z_1, T_1 are four new linear functions of x, y, z, whose coefficients as well as the exponents r and s are determined by the coefficients of X, Y, Z, T.

1) *Halphen,* Acta Mathematica, vol. 3.
2) *Fouret,* Comptes Rendus. October 1876.

To prove this theorem, consider the partial differential equation

$$(43) \qquad X\frac{\partial z}{\partial x} + Y\frac{\partial z}{\partial y} - Z + T\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right) = 0.$$

The ordinary method for integrating it consists in setting up precisely the system of ordinary differential equations (41). The factors of X, Y, Z, T in (43), are the coordinates of the plane tangent to a surface $z = f(x, y)$ at the point x, y, z. Denote them by ξ, η, ζ, τ, so that we have an equation of the form

$$X\xi + Y\eta + Z\zeta + T\tau = 0$$

According to *Clebsch*, such an equation containing both point- and plane-coordinates, defines a *connex* of space, i. e. the totality of points (m) and planes (μ) which satisfy the equation. The *order* and *class* of the connex are the degree of the equation with respect to the point- and the plane-coordinates respectively. In our case, the order and class are both equal to unity. To every point m of space, there corresponds an infinity of planes μ; but, as the equation of the connex shows, all of these planes μ pass through a point m'. The relation between m and m' is projective. There will exist, in general, a definite tetrahedron whose vertices correspond to themselves in this projective relation. If this be taken as tetrahedron of reference, the equation of the connex becomes

$$AX_1\xi_1 + BY_1\eta_1 + CZ_1\zeta_1 + DT_1\tau_1 = 0,$$

where X_1, Y_1, Z_1, T_1 are the new homogeneous coordinates of m, and where $\xi_1, \eta_1, \zeta_1, \tau_1$ are those of μ. Let us return to non-homogeneous coordinates by putting

$$\frac{X_1}{T_1} = x_1, \quad \frac{Y_1}{T_1} = y_1, \quad \frac{Z_1}{T_1} = z_1,$$

so that the equation of the connex becomes

$$Ax_1\xi_1 + By_1\eta_1 + Cz_1\zeta_1 + D\tau_1 = 0.$$

The partial differential equation (43) requires of all of its integral surfaces: in each point m of such a surface, the tangent plane shall be one of those which corresponds to m in the connex Consequently, in the new variables, this partial differential equation will assume the form

$$Ax_1\frac{\partial z_1}{\partial x_1} + By_1\frac{\partial z_1}{\partial y_1} - Cz_1 + D\left(z_1 - x_1\frac{\partial z_1}{\partial x_1} - y_1\frac{\partial z_1}{\partial y_1}\right) = 0.$$

But this equation is integrated by means of the auxiliary system

$$(44) \qquad \frac{dx_1}{(A-D)x_1} = \frac{dy_1}{(B-D)y_1} = \frac{dz_1}{(C-D)z_1}.$$

In other words, there exists a system of four polynomials X_1, Y_1, Z_1, T_1 such that the system (41) assumes the form (44) if $X_1, \ldots T_1$ be taken as new variables.

But (44) may be integrated at once. We find

$$y_1 = cx_1^r, \quad z_1 = c' x_1^s,$$

or

$$T_1^{r-1} Y_1 = c X_1^r, \quad T_1^{s-1} Z_1 = c' X_1^s, \tag{45}$$

where

$$r = \frac{B-D}{A-D}, \quad s = \frac{C-D}{A-D},$$

which completes the proof of the theorem.

We see, therefore, that the equations (41) may be taken as the differential equations of an anharmonic curve. If the polynomials X, Y, Z, T are given, the determination of the polynomials X_1, Y_1, Z_1, T_1 accomplishes the solution of our problem: to find the principal tetrahedron of the anharmonic.

Equations (45) clearly contain 16 arbitrary constants. For, we may take the constant terms in the four polynomials X_1, Y_1, Z_1, T_1 equal to unity. The twelve coefficients which remain in these polynomials, the exponents r and s, and the constants c and c' constitute, in fact, 16 arbitrary constants. If we wish to determine the osculating anharmonic of the given curve at a given point, we shall have to determine these sixteen constants in such a way that the contact between the two curves shall be of the seventh order, which gives precisely sixteen conditions. These conditions might be written down by differentiating (45) seven times, and substituting into the resulting equations, for the derivatives up to the seventh order, the numerical values of the corresponding derivatives for the given curve. But, equations (41) clearly represent the result of eliminating the two constants c and c' from these sixteen conditions, and, in fact, they contain only fourteen constants. These fourteen constants may, therefore, be determined by means of (41) and those obtained therefrom by six-fold differentiation. After they have been computed c and c' are easily obtained by the condition that the anharmonic curve must pass through the given point of the given curve. All of the conditions, obtained in this way for the sixteen constants, are linear, so that a unique solution exists.

We saw, in the proof of *Fouret*'s theorem, how a certain connex was related to the problem of finding the principal tetrahedron of the osculating anharmonic. This connex is not completely determined. In fact, if we add to the left member of its equation

$$X\xi + Y\eta + Z\zeta + T\tau = 0,$$

any numerical multiple of $x\xi + y\eta + z\zeta + t\tau$, its tetrahedron does not change. We may take advantage of this fact. It enables us to

annul the constant term in T. We shall assume, therefore, that this constant term is zero.

We now write equations (41) as follows

$$Xy' - Y - T(xy' - y) = 0, \quad Xz' - Z - T(xz' - z) = 0, \tag{46}$$

and assume that the equations of the given curve are given in their canonical form

$$\begin{aligned} z &= x^3 + \alpha_6 x^6 + \alpha_7 x^7 + \quad, \\ y &= x^2 + \beta_7 x^7 + \quad. \end{aligned} \tag{47}$$

In order that (46) may be the differential equations of *that* anharmonic curve, which osculates the curve (47) at the origin, equations (47) must satisfy (46) up to and including terms of the sixth order. In this way the polynomials $X, \ldots T$ are determined. We find

$$\begin{aligned} X &= 7s_7 x - 42\frac{t_7}{v}y - 18v^2 z - 3v, \\ Y &= -6vx + 14s_7 y - 21\frac{t_7}{v}z, \\ Z &= -9vy + 21s_7 z, \\ T &= -63\frac{t_7}{v}x - 36v^2 z. \end{aligned} \tag{48}$$

The explicit determination of the polynomials $X_1, \ldots T_1$, i. e. of the principal tetrahedron of the osculating anharmonic, requires the solution of an equation of the fourth degree. The general theory of collineations furnishes all of the material for further discussion of this problem, a discussion which we shall not, however, undertake.

§ 4. Relation to the theory of plane curves.

We shall consider briefly, two plane curves which are determined by the properties of a space curve (m) in the vicinity of one of its points m. Let us take m as a center and project the curve (m) from m upon any plane. The curve (M), which is obtained in this way, is the first of the two plane curves which we wish to consider. The other curve (M_1) is obtained by a construction dualistic to this, viz.: it is the intersection of the developable, whose cuspidal edge is the given curve (m), with the plane osculating (m) at m.

Let the equations of (m) be developed in their canonical form:

$$\begin{aligned} y &= x^2 + \beta_7 x^7 + \beta_8 x^8 + \cdots, \\ z &= x^3 + \alpha_6 x^6 + \alpha_7 x^7 + \alpha_8 x^8 + \cdots, \end{aligned} \tag{49}$$

where

$$\begin{aligned} \alpha_6 &= v, \ \alpha_7 = s_7, \quad \alpha_8 = \frac{1}{v}\left(s_8 + 3t_7 + \frac{7}{6}s_7^2\right), \\ \beta_7 &= \frac{1}{v}t_7, \qquad \beta_8 = \frac{1}{v^2}\left(t_8 + \frac{7}{3}s_7 t_7 - \frac{3}{2}v^4\right). \end{aligned} \tag{50}$$

Using the origin as center, and the plane at infinity as plane of projection, we may take

$$X = \frac{y}{x}, \quad Y = \frac{1}{2}\frac{z}{x},$$

as the coordinates of a point M of the curve (M). We shall then find

$$Y = \frac{1}{2} X^2 + \frac{1}{2} \alpha_6 X^5 + \frac{1}{2} \alpha_7 X^6 + \frac{1}{2} (\alpha_8 - 2\beta_7) X^7 + \cdots. \tag{51}$$

The factor 2 has been introduced for convenience, enabling us to compare, more immediately, this form of the development with the canonical development for a plane curve (Chapter III, § 7).

If (51) be reduced to the canonical form by the method of Chapter III, § 7, and the new coordinates be still denoted by X, Y, the following development will be obtained,

$$Y = \frac{1}{2} X^2 + X^5 + A_7 X^7 + \cdots,$$

where

$$A_7 = \frac{1}{2^{1/3} v^{8/3}} \left(s_8 + t_7 + \frac{1}{6} s_7^2 \right) - \frac{\delta}{2^{1/3} v^{8/3}}. \tag{52}$$

The canonical form for the second plane curve (M_1), mentioned before, will be obtained by putting in A_7 for each invariant its adjoint, since the two curves are dualistic to each other. This may be easily carried out, since we have the expressions for v, s_7, t_7, s_8, t_8 in terms of Θ_3, Θ_4, etc.[1]), and since the adjoints of Θ_3 and Θ_4 are $-\Theta_3$ and Θ_4 respectively. Let A_7 be the value of the coefficient thus obtained Then

$$A_7 = \frac{1}{2^{1/3} v^{8/3}} \left(\frac{5}{3} s_8 + t_7 + \frac{1}{6} s_7^2 \right) - \frac{\delta_1}{2^{1/3} v^{8/3}}. \tag{53}$$

If $\delta = 0$, the origin is a *coincidence* point upon (M)[2]) We may express this as follows. *We may construct cubic cones having the point m of the curve (m) as vertex, and the tangent to (m) as double generator. It is possible to determine such a cone having twelve coincident points of intersection with the curve at the point m, if and only if the invariant δ is equal to zero.*

Another proof of this may be easily obtained as follows. From (49) we find

$$\alpha_6 (xyz - y^3 - \alpha_6 z^3) - \alpha_7 (xz^2 - y^2 z) = [\alpha_6 (\alpha_8 - 2\beta_7) - \alpha_7^2] x^{11} + \cdots. \tag{54}$$

The left member, equated to zero, is the equation of a cubic cone of the character described. It has twelve coincident points of inter-

1) Equations (21).
2) Cf. Chapter III, § 4.

section with (m) at the origin, if the coefficient of x^{11} on the right member vanishes. But this coefficient may be easily shown to be equal to δ. In general, this cubic cone has only eleven coincident points of intersection with (m) at the origin.

By duality we find the further result. *The plane curve, in which the osculating plane of the point m intersects the developable, of which the curve (m) is the cuspidal edge, has m as a point of coincidence, if and only if $\delta_1 = 0$.*

If Θ_3 and Θ_8 are the invariants of the *plane* curve (M), we have

$$A_7 = \frac{(-20)^{8/3}}{100800} \frac{\Theta_8}{\Theta_3^{8/3}}.$$

Let R be the exponent of the osculating anharmonic of (M) at M. Then

$$\frac{\Theta_8^3}{\Theta_3^8} = 3^9 \frac{(R^2 - R + 1)^2}{(R-2)^2 (1-2R)^2 (R+1)^2} \; {}^{1)},$$

so that the exponent R is determined by the equation

$$\frac{\left(s_8 + t_7 + \frac{1}{6} s_7^2\right)^3}{v^8} = \frac{3^3 \, 5^2}{2^4 \cdot 7} \frac{(R^2 - R + 1)^2}{(R-2)^2 (1-2R)^2 (R+1)^2}. \tag{55}$$

There will be a similar equation for R_1. Comparing the two equations we find:

The exponent R_1 of the anharmonic, which osculates the second plane curve (M_1) at M_1, will coincide with R if $s_8 = 0$.

If the given curve (m) is an anharmonic, s_8 is identically equal to zero, so that R and R_1 are equal to each other at all points of the curve. Moreover, (55) is the equation for determining R. The left member of (55) may be expressed in terms of the exponents λ and μ of the curve (m) which may be supposed to have the equations

$$y = x^\lambda \quad z = x^\mu . {}^{2)}$$

These considerations lead *Halphen* to make the following remark. A space anharmonic curve (m) with given invariants λ and μ, gives rise in each of its points to a perspective (M), which is not a plane anharmonic The plane anharmonic (M'), which osculates (M) at the point M which corresponds to m, has for its exponent the quantity R given by equation (55). This is true for all general positions of the point m, but ceases to be true when m coincides

1) Chapter III, equ. (60).

2) For the purpose of computing the left member of (55) in terms of λ and μ we have the equations (37).

with one of the vertices of the fundamental tetrahedron of the curve (m). In fact, let $\lambda > 0$, $\mu > 0$. Then the curve

$$y = x^{\lambda}, \quad z = x^{\mu}$$

passes through the origin. The cone, having the origin as vertex, and the curve as directrix, will be

$$z^{\lambda - 1} = x^{\lambda - \mu} y^{\mu - 1}.$$

The plane sections of this cone are anharmonic curves whose exponent may be equated to $\frac{\lambda - \mu}{\lambda - 1}$. If this value be substituted for R in (55), the left member, moreover, being expressed in terms of λ and μ, the equation is not, in general, satisfied.

Therefore, if the point m moves into one of the singular points of the anharmonic curve (m), the perspective (M) becomes an anharmonic curve, but its invariant ceases to be equal to R.

§ 5. Some applications to the theory of ruled surfaces.

The principal surface of the flecnode congruence was defined in a somewhat unsatisfactory manner in the following exceptional cases; 1st when the ruled surface has two rectilinear directrices; 2d when the two branches of the flecnode curve coincide. We are now in a position to simplify these definitions considerably.

Consider the case that the ruled surface S has two distinct rectilinear directrices. Let C_y and C_z be two curved asymptotic lines upon S. We may then assume

$$p_{ik} = 0, \quad q_{12} = aq, \quad q_{21} = bq, \quad q_{11} - q_{22} = cq. \tag{56}$$

Let us form the differential equation of the fourth order for C_y, according to the formulae of Chapter XII.

We find:

$$\begin{aligned} y^{(4)} - 2\frac{q'}{q} y^{(3)} + \left[q_{11} + q_{22} - \frac{q''}{q} + 2\left(\frac{q'}{q}\right)^2\right] y'' - 2\left(q_{11}\frac{q'}{q} - q_{11}'\right) y' \\ - \left[- q_{11}(q_{11} + q_{22}) + q_{11}\left\{\frac{q''}{q} - 2\left(\frac{q'}{q}\right)^2\right\}\right. \\ \left. + 2q_{11}'\frac{q'}{q} + q_{11}^2 + abq^2 - q_{11}''\right] y = 0. \end{aligned} \tag{57}$$

The derivative curve of the first kind, (upon the developable of C_y), is given by the expression

$$y' - \frac{1}{2}\frac{q'}{q} y$$

Similarly, the derivative of the first kind of C_z is given by

$$z' - \frac{1}{2}\frac{q'}{q}z.$$

On the other hand, the generator of the ruled surface S', the derivative of S with respect to x, joins the two points

$$y' \text{ and } z'.$$

These are the same as the above, if and only if $q' = 0$, i. e. if Θ_4 is a constant, i. e. if S' is the principal surface of the congruence.

We see, therefore, that the following theorem is true.

Let S be a ruled surface with rectilinear directrices, defined by a system of form (A). *Its derivative S', with respect to x, contains the derivatives of the first kind of the asymptotic curves of S, if and only if S' is the principal surface of the flecnode congruence.*

Precisely the same property will be found to characterize the principal surface in the case that the two sheets of the flecnode surface coincide.

In (56) we may assume $c = 0$. C_y and C_z will then be two asymptotic curves, whose intersections with any generator are harmonic conjugates of each other, with respect to the two points in which the generator intersects the directrices of the surface. The differential equations for C_y and C_z become identical, i. e.: *two asymptotic curves which are so related, are projectively equivalent.* But these two curves are really two parts of one irreducible curve. For, S belongs to an infinity of linear complexes, and the complex curve of S, with respect to each of these complexes, is precisely such an asymptotic curve, each of which intersects every generator twice in the above fashion. We may, therefore, say that *every asymptotic curve of a ruled surface with rectilinear directrices admits of projective transformation into itself.*

Each of these asymptotic lines obviously belongs to a linear complex. This may, moreover, be easily verified from equation (57), whose invariant of weight 3 is zero. But the converse is also true, i. e.: *if the asymptotic curves of a ruled surface belong to linear complexes, the surface must have two rectilinear directrices.* This theorem is due to *Peters.*[1]) We have developed all of the formulae necessary for its proof, the details of which we shall leave to the reader.

1) *Peters.* Die Flächen, deren Haupttangentenkurven linearen Komplexen angehören. (Leipzig, Dissertation 1895.) Christiania und Kopenhagen. Alb. Cammermeyers Forlag.

§ 6. On the order of contact between curves after a dualistic transformation.

If two curves (m) and (m') have a contact of the n^{th} order at a common point m, two curves (M) and (M') obtained from them by a dualistic transformation will, in general, have contact of a different order. The following investigation, which is due to *Halphen*, will make clear the various cases which may arise.

Since the most general dualistic transformation may be obtained by combining any special dualistic transformation with a general projective transformation, and since the latter leaves the order of contact invariant, we may confine ourselves to investigating the effect of any particular dualistic transformation.

Let x, y, z, t be the homogeneous coordinates of a point m of the curve (m). We may take as the homogeneous coordinates of a point M of the curve (M), which is dualistic to (m), the four determinants of the third order

$$(tx'y''), \quad (zt'x''), \quad (yz't''), \quad (xy'z''),$$

which are proportional to solutions of the *Lagrange* adjoint of the differential equation of the curve (m) To return to non-homogeneous coordinates, we put $t = 1$, take x as the independent variable, and consider the ratios of the above four determinants The coordinates X, Y, Z of the point M, which is thus made to correspond dualistically to the point m, will be

$$\begin{aligned} X &= \frac{(zt'x'')}{(tx'y'')} = \frac{z''}{y''}, \\ Y &= \frac{(yz't'')}{(tx'y'')} = \frac{y'z'' - z'y''}{y''}, \\ Z &= \frac{(xy'z'')}{(tx'y'')} = \frac{x(y'z'' - z'y'') + zy'' - yz''}{y''}. \end{aligned} \tag{58}$$

These relations between X, Y, Z and x, y, z are reciprocal and must, therefore, give rise to reciprocal relations between the elements of the two curves.

Let m be chosen as origin of coordinates, the tangent to the curve at that point as x axis, and the osculating plane as the plane $z = 0$. The same conditions will then be fulfilled for the curve (M) at M. The development of the equations of the curve (m) in the vicinity of m will be of the form

$$y = y_2 x^2 + y_3 x^3 + \cdots,$$
(59)
$$z = z_3 x^3 + z_4 x^4 + \cdots,$$

and similarly for (M)

$$Y = Y_2 X^2 + Y_3 X^3 + \cdots,$$
(60)
$$Z = Z_3 X^3 + Z_4 X^4 + \cdots.$$

We proceed to find the relations between the coefficients of these two expansions, by making use of (58). For this purpose, we shall develop both members of each of the equations (60) into series proceeding according to powers of x, and then identify the coefficients of like powers.

Let us denote by π_n a function of the quantities y_m and z_m in which the indices do not exceed n. Whenever it is not necessary to distinguish between two such functions, the same letter π_n may be used for both.

From (59) we find

(61)
$$\frac{(z'')^m}{(y'')^{m-1}} = \left(\frac{3z_3}{y_2}\right)^m 2y_2 x^m \left\{1 + \left[2m\frac{z_4}{z_3} - 3(m-1)\frac{y_3}{y_2}\right]x + \cdots \right.$$
$$\left. + \left[\frac{n(n+1)}{2\cdot 3}m\frac{z_{n+1}}{z_3} + \pi_n\right]x^{n-2} + \cdots\right\},$$
$$y'z'' - z'y'' = 2y_2 x^2 \{3z_3 + 8z_4 x + \cdots$$
$$+ [(n^2-1)z_{n+1} + \pi_n]x^{n-2} + \cdots\}$$

Owing to (58), the first equation of (60) may be written

$$y'z'' - z'y'' = Y_2\frac{(z'')^2}{y''} + Y_3\frac{(z'')^3}{(y'')^2} + \cdots.$$

If we put

(63)
$$G = \frac{3z_3}{y_2}, \quad g = \frac{1}{G},$$

the substitution of (61) in (62) gives the following equations:

$$3z_3 = G^2 Y_2,$$
$$8z_4 = G^2\left(4\frac{z_4}{z_3} - 3\frac{y_3}{y_2}\right)Y_2 + G^3 Y_3,$$
$$\cdots\cdots\cdots\cdots$$
$$\pi_n + (n^2-1)z_{n+1} = G^2\left[\frac{n(n+1)}{3}\frac{z_{n+1}}{z_3} + \pi_n\right]Y_2 + \pi_n Y_3 + \pi_{n-1}Y_4 + \cdots$$
$$+ \pi_4 Y_{n-1} + G^n Y_n,$$

whence

$$
(64) \qquad \begin{aligned}
Y_2 &= g y_2, \\
Y_3 &= -4g^3 z_4 + 3g^2 y_3, \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
Y_n &= -g^n (n+1) z_{n+1} + \pi_n.
\end{aligned}
$$

We have further

$$
\begin{aligned}
&x(y'z'' - z'y'') + zy'' - yz'' \\
&\quad = 2y_2 x^3 \left\{ z_3 + 3z_4 x + \cdots + \left[\frac{n(n-1)}{2} z_{n+1} + \pi_n\right] x^{n-2} + \cdots \right\}.
\end{aligned}
$$

Making use of this formula, the second equation of (60) gives:

$$
(65) \qquad \begin{aligned}
Z_3 &= g^3 z_3, \\
Z_4 &= -3g^4 z_4 + 2g^3 y_3, \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
Z_n &= -g^n (n-1) z_n + \sigma_{n-1},
\end{aligned}
$$

where σ_{n-1} depends upon quantities whose index is no higher than $n-1$.

The quantities G and g are dualistic. In fact we find

$$
\frac{3Z_3}{Y_2} = 3g^2 \frac{z_3}{y_2} = g^2 G = g.
$$

The equations (64) and (65) must be self-dual, i. e. they must remain true if the small and capital letters are interchanged. That this is so may be easily verified for the first two equations of each system, these equations having been completely determined. On the other hand, this dualistic character of the equations enables us to supply some further terms in the expressions for Y_n and Z_n.

In fact, let us interchange the small and capital letters in the equation for Y_n, and denote by Π_n the new value of π_n. Then, if we substitute for Z_{n+1} its value, we shall find

$$
y_n = n(n+1) g z_{n+1} - \frac{1}{g^n} \sigma_n + \Pi_n,
$$

where σ_n depends upon the quantities y_m and z_m of index no higher than n. This equation must become an identity, if we replace, in Π_n, the quantities Y_m, Z_m by their expressions in terms of y_m, z_m. This substitution in Π_n must, therefore, give rise to a term $-n(n+1) g z_{n+1}$. But (64) and (65) show that only the expression $+nG^{n-1} Y_n$ can give rise to such a term. We have, therefore,

$$
(66) \qquad Y_n = -(n+1) g^n z_{n+1} + n g^{n-1} y_n + [z_n, y_{n-1}],
$$

where $[z_n, y_{n-1}]$ contains the quantities z_m whose index is no higher than n, and the quantities y_m whose index does not exceed $n-1$.

Similarly we find

$$(67) \quad Z_n = -(n-1)g^n z_n + (n-2)g^{n-1}y_{n-1} + [z_{n-1}, y_{n-2}].$$

The equations (66) and (67) enable us to solve the proposed problem. Consider two curves (m) and (m') which have a contact of the n^{th} order at the origin. In the vicinity of the origin, the curve (m) is represented by the equations (59), and the curve (m') by two equations of the same form. We shall then have, by hypothesis,

$$y_2' = y_2, \quad y_3' = y_3, \quad \ldots \quad y_n' = y_n,$$
$$z_3' = z_3, \quad \ldots \quad z_n' = z_n.$$

Equations (66) and (67) show that, for the two reciprocal curves (M) and (M'), we shall have

$$Y_2' = Y_2, \quad Y_3' = Y_3, \quad \ldots \quad Y_{n-1}' = Y_{n-1},$$
$$Z_3' = Z_3, \quad \ldots \quad Z_{n-1}' = Z_{n-1}, \quad Z_n' = Z_n,$$

so that the order of their contact will, in general, be only $n-1$. But it becomes necessary to examine the conditions for this more carefully. For, the same transformation will convert (M) and (M') back into (m) and (m'), so that the order of contact may be increased, as well as diminished, by a dualistic transformation.

We notice in the first place that the curves (M) and (M') satisfy the condition $Z_n' = Z_n$, i. e. if the order of their contact is really $n-1$, their principal tangent plane coincides with the osculating plane.[1]) Further we notice that

$$T_{n+1} = 3nZ_3Y_n - (n+1)Y_2Z_{n+1},$$

when expressed in terms of y_m, z_m contains none of these quantities of index higher than n. The curves (M) and (M'), therefore, satisfy the further relation $T_{n+1}' = T_{n+1}$.

We may now state our result as follows:

1. *If two curves have, at the point m, a contact of the n^{th} order, and if their principal tangent plane at this point does not coincide with the osculating plane, the two curves which correspond to them in any dualistic transformation will have, at the corresponding point, a contact of the $n-1^{\text{th}}$ order only.*

2. *If the principal tangent plane does coincide with the osculating plane, but if the function*

1) Chapter XIII, § 7.

$$t_{n+2} = 3(n+1)z_3 y_{n+1} - (n+2)y_2 z_{n+2}$$

has different values for the two curves, the order of contact is not changed by the dualistic transformation.

3. *If the principal tangent plane coincides with the osculating plane, and if, besides, the function t_{n+2} has the same value for both curves, the dualistic transformation increases the order of contact to $n+1$.*

This theorem makes it evident that, to the canonical development of the curve (m) will not correspond, in general, the canonical development of the curve (M). For, the osculating cubic of (m) is not, in general, transformed by a dualistic transformation into the osculating cubic of (M).

It need scarcely be remarked that our proof presupposes $n \geqq 2$, so that the theorem does not contradict the fact that a dualistic transformation always converts a pair of curves which touch into another pair which, likewise, touch.

Examples.

Ex. 1. In terms of *Halphen*'s canonical invariants, find the condition for a curve on a quadric surface. Express this in the usual invariants.

Ex. 2. Transform the figure composed of a curve and its osculating anharmonic dualistically The result of this transformation is a curve and its osculating anharmonic, if and only if $s_8 = 0$. *(Halphen.)*

Ex. 3. If $s_8 = 0$, the curve and its osculating anharmonic have the osculating plane for their principal tangent plane If $t_8 = 0$, the principal tangent plane of these two curves coincides with the principal plane of the curve and its osculating cubic. *(Halphen.)*

Ex. 4. If the curve (m) belongs to a linear complex, the osculating conic of the plane curve (M) of § 4 hyperosculates it. What is the corresponding property of the second plane curve (M_1) of § 4? *(Halphen.)*

Ex. 5. Find the condition for a curve on a quadric cone. *(Halphen.)*

Ex. 6. A biquadratic curve may be obtained which has contact of the seventh order with (m) at a given point (osculating biquadratic). Its developments will be

$$z = x^3 + vx^6 + s_7 x^7 + \frac{1}{v}(t_7 + s_7^2)x^8 + \cdots,$$

$$y = x^2 + \frac{1}{v} t_7 x^7 + \frac{1}{v^2} s_7 t_7 x^8 + \cdots.$$

Consider the following four planes through the tangent of (m); the osculating plane, the principal plane of (m) and its osculating cubic,

the principal plane of (m) and its osculating anharmonic, the principal plane of (m) and its osculating biquadratic. Their anharmonic ratio is

$$\frac{s_8}{t_8} - \frac{t_8 + \frac{4}{3} s_7 t_7 - \frac{3}{2} v^4}{s_8 + 2t_7 + \frac{1}{6} s_7^2}. \quad \textit{(Halphen.)}$$

Ex. 7. The conditions

$$\frac{13}{3} s_8 + 2t_7 + \frac{1}{6} s_7^2 = 0, \quad \frac{8}{3} s_9 + t_8 - \frac{32}{9} s_7 s_8 - \frac{4}{3} s_7 t_7 + \frac{3}{2} v^4 = 0$$

characterize those curves whose developables are circumscribed about two quadric surfaces. *(Halphen.)*

CORRECTIONS.

Page	19,	line 5	from bottom	read	k	instead of	λ
„	80,	„ 10	„ top	„	as many	„	a many.
„	109,	„ 8	„ „	„	$\bar{\alpha}$	„	α
„	163,	„ 7	„ „	„	a_{13}, b_{13}	„	a_{31}, b_{31}.

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THE REFERENCES ARE TO PAGES.